ORF 307: Lecture 14

Linear Programming: Chapter 14: Network Flows: Algorithms

Robert J. Vanderbei

April 9, 2019

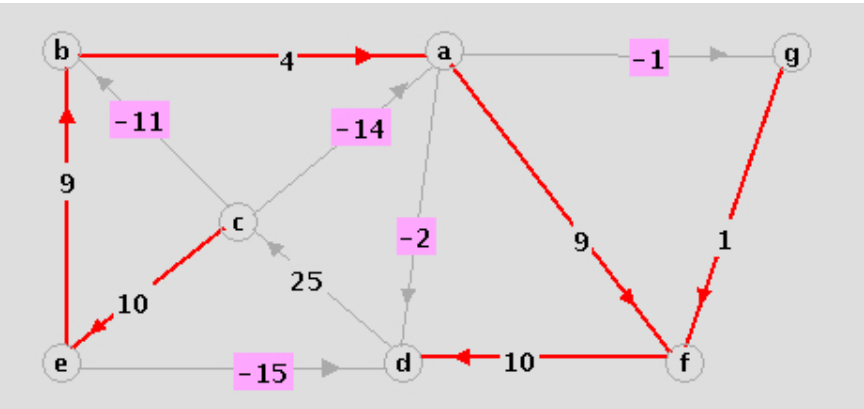
Slides last edited on January 25, 2019

Agenda

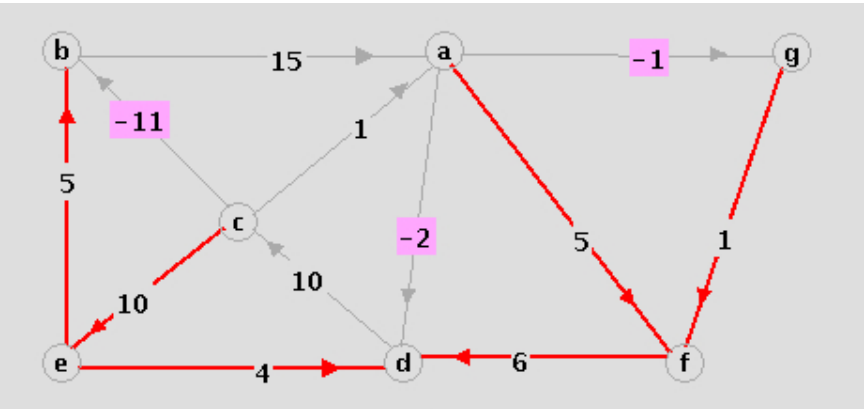
- Primal Network Simplex Method
- Dual Network Simplex Method
- Two-Phase Network Simplex Method
- One-Phase Primal-Dual Network Simplex Method
- Planar Graphs
- Integrality Theorem

Primal Network Simplex Method

Used when all primal flows are nonnegative (i.e., primal feasible).



Entering arc: (e,d)
Leaving arc: (b,a)

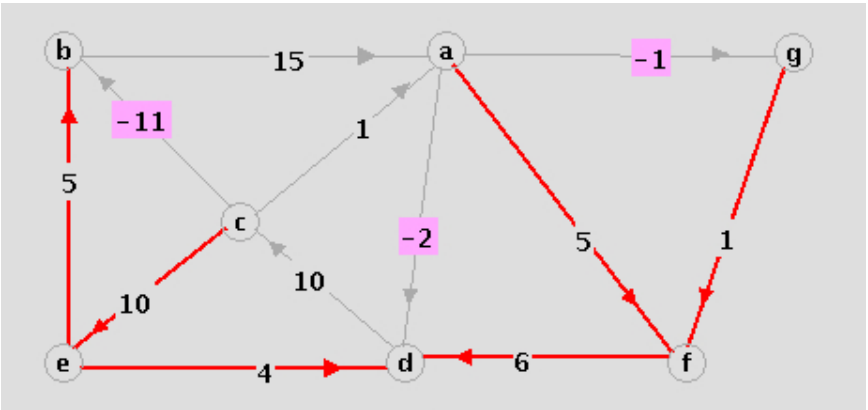


Pivot Rules:

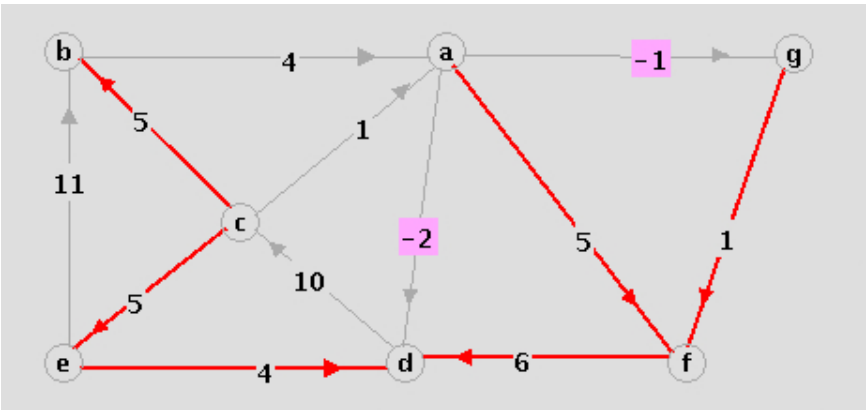
Entering arc: Pick a nontree arc having a negative (i.e. infeasible) dual slack.

Leaving arc: Add entering arc to make a *cycle*.
Leaving arc is an arc on the cycle, pointing in the *opposite* direction to the entering arc, and
of all such arcs, it is the one with the *smallest* primal flow.

Primal Method—Second Pivot



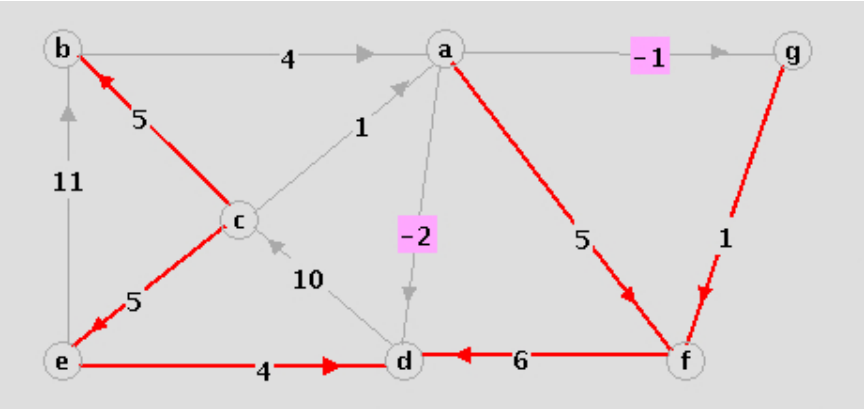
Entering arc: (c,b)
Leaving arc: (e,b)



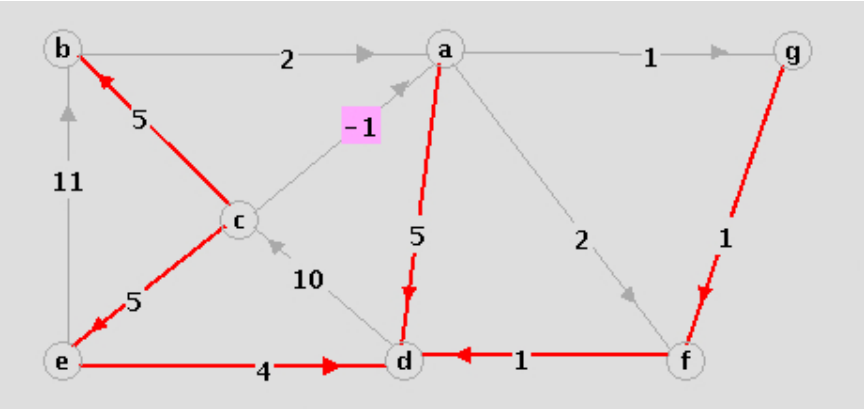
Explanation of leaving arc rule:

- Increase flow on (c,b).
- Each unit increase produces a unit *increase* on arcs pointing in the *same* direction.
- Each unit increase produces a unit *decrease* on arcs pointing in the *opposite* direction.
- The first to reach zero will be the one pointing in the opposite direction and having the smallest flow among all such arcs.

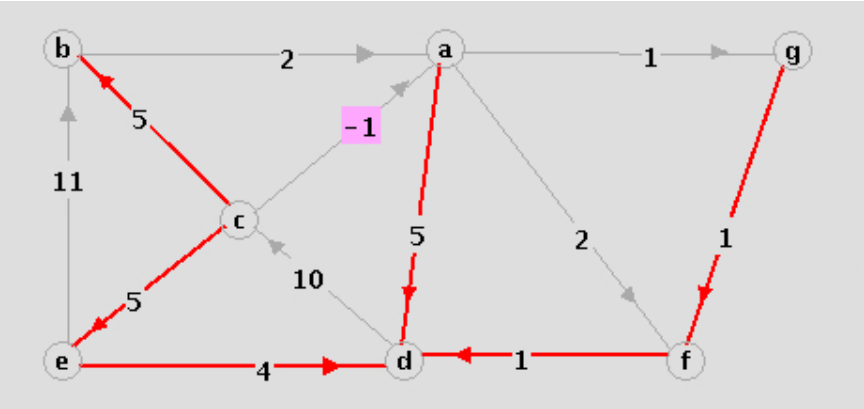
Primal Method—Third Pivot



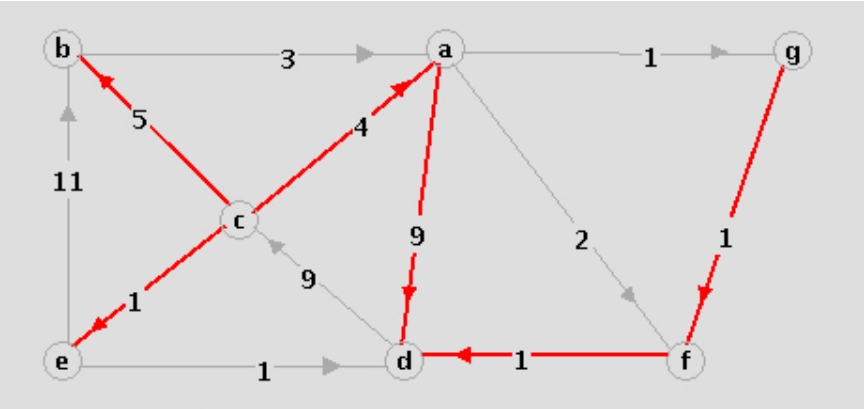
Entering arc: (a,d)
Leaving arc: (a,f)



Primal Method—Fourth Pivot



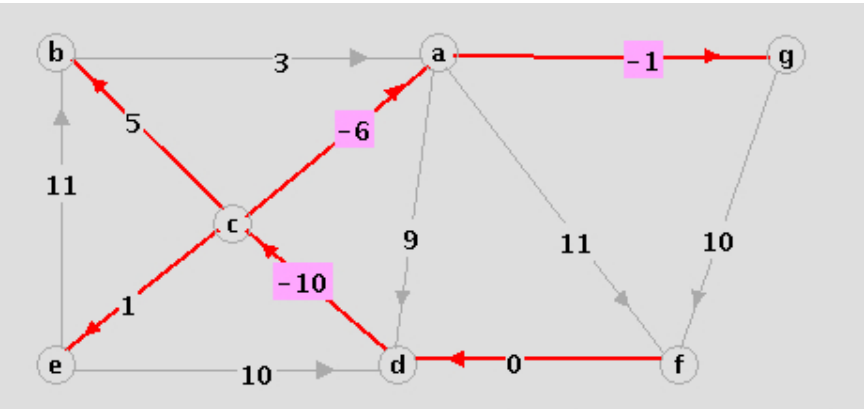
Entering arc: (c,a)
Leaving arc: (e,d)



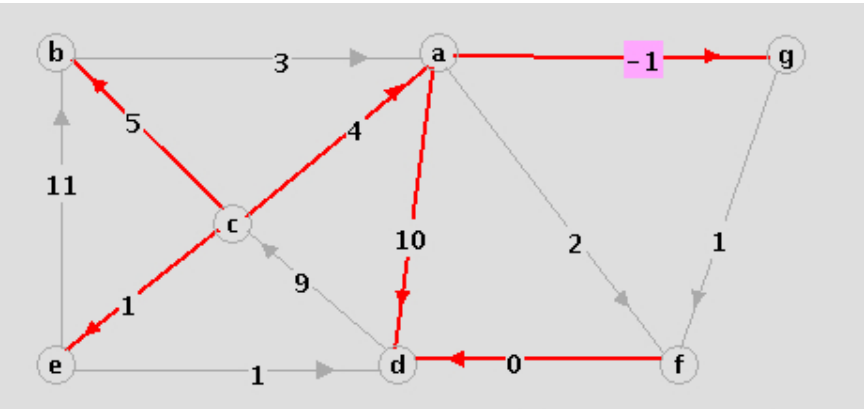
Optimal!

Dual Network Simplex Method

Used when all dual slacks are nonnegative (i.e., dual feasible).



Leaving arc: (d,c)
Entering arc: (a,d)

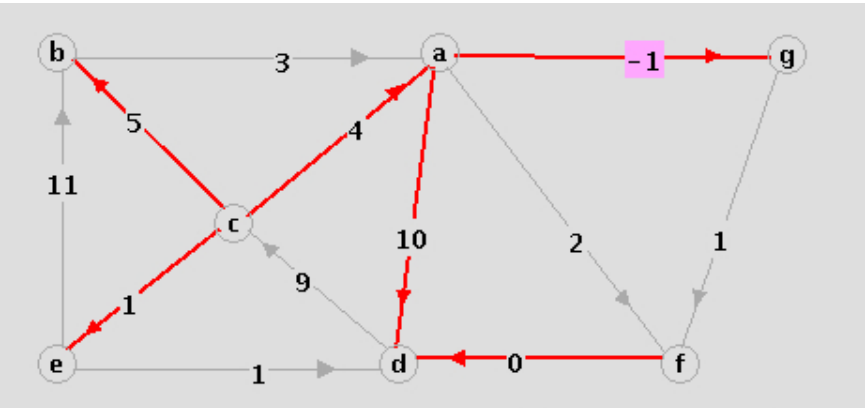


Pivot Rules:

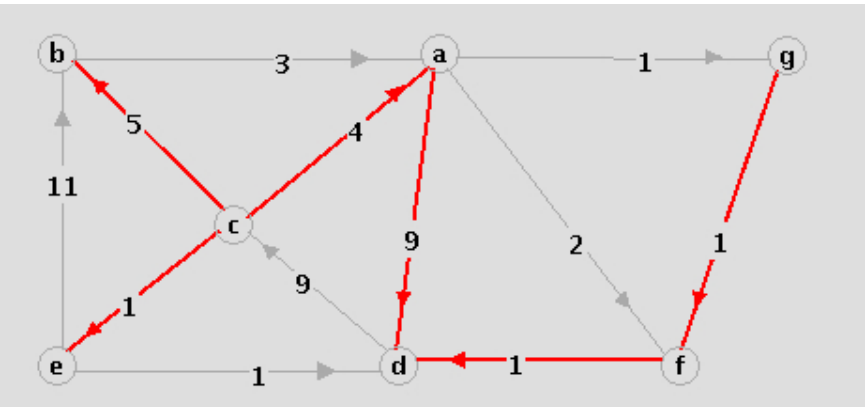
Leaving arc: Pick a tree arc having a negative (i.e. infeasible) primal flow.

Entering arc: Remove leaving arc to split the spanning tree into two subtrees. Entering arc is an arc reconnecting the spanning tree with an arc in the *opposite* direction, and, of all such arcs, is the one with the *smallest* dual slack.

Dual Network Simplex Method—Second Pivot



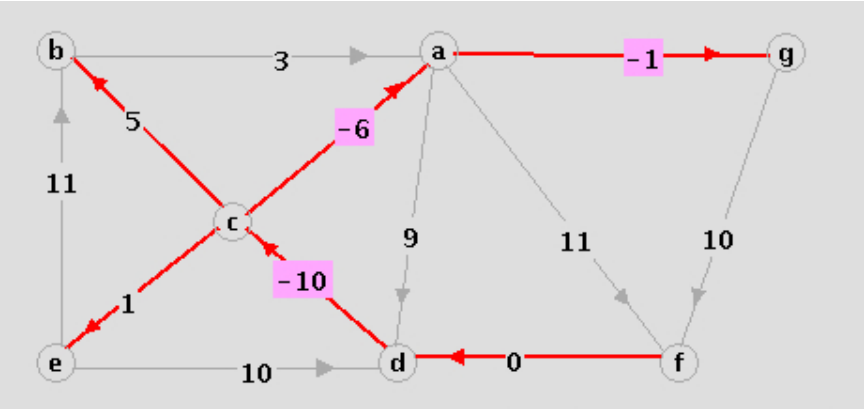
Leaving arc: (a,g)
Entering arc: (g,f)



Optimal!

Explanation of Entering Arc Rule

Recall initial tree solution:



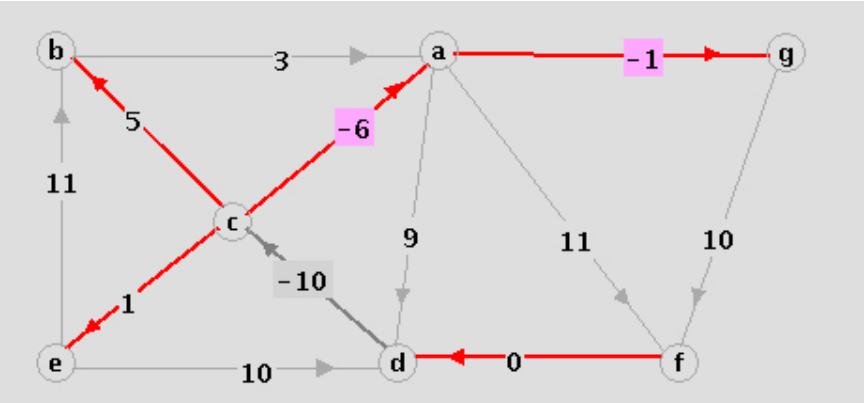
Leaving arc: (d,c)
Entering arc: (a,d)

- Remove leaving arc. Need to find a reconnecting arc.
- Since the leaving arc has a negative flow, there is a net supply at the subtree attached to the head node and a net demand at the subtree attached to the tail node.
- So, reconnecting with an arc that spans in the same direction does not improve anything.
- Hence, only consider arcs spanning the two subtrees in the opposite direction.

- Consider a potential arc reconnecting in the opposite direction, say (a,f).
  - Its dual slack will drop to zero.
  - All other reconnecting arcs pointing in the same direction will drop by the same amount.
  - To maintain nonnegativity of all the others, must pick the one that drops the least.

Explanation of Entering Arc Rule

Recall initial tree solution:



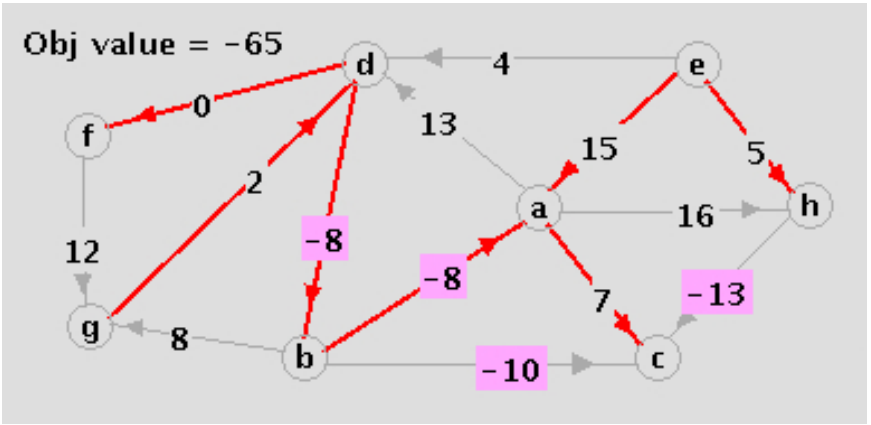
Leaving arc: (d,c)
Entering arc: (a,d)

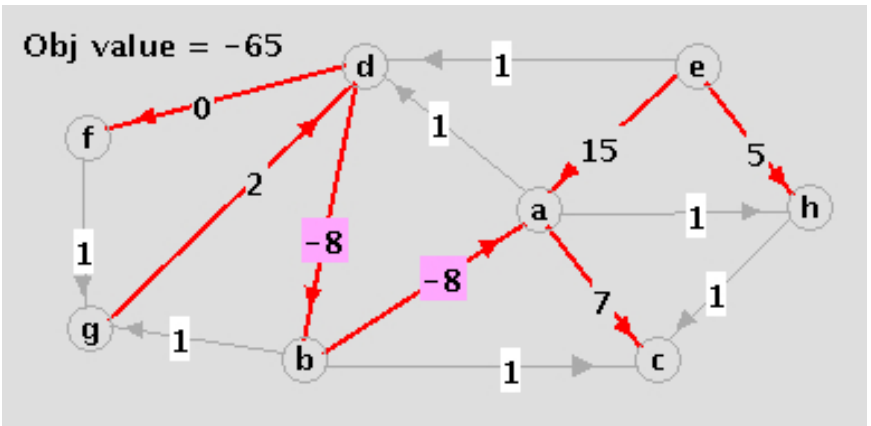
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Two-Phase Network Simplex Method

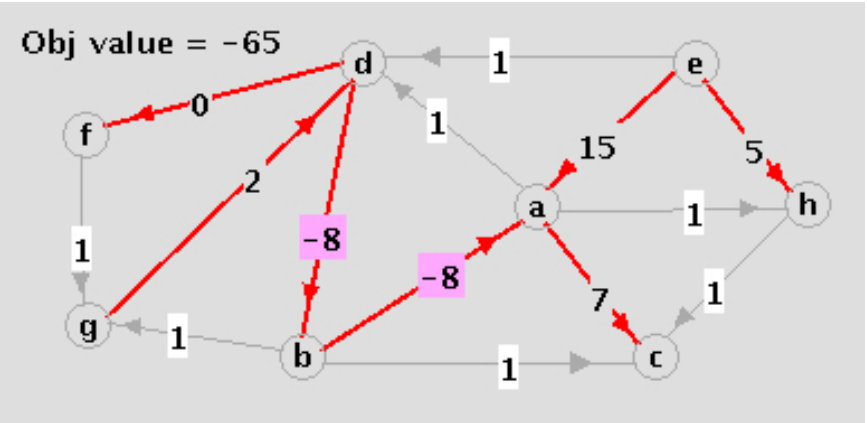
Example.



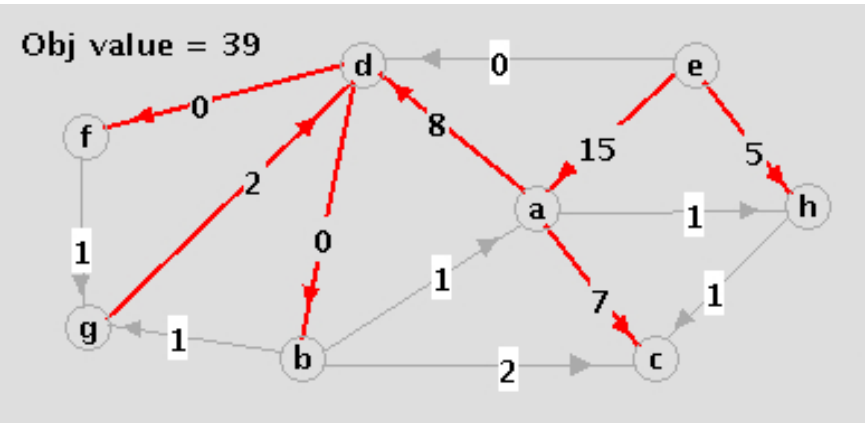


- Turn off display of dual slacks.
- Turn on display of artificial dual slacks.

Two-Phase Method–First Pivot



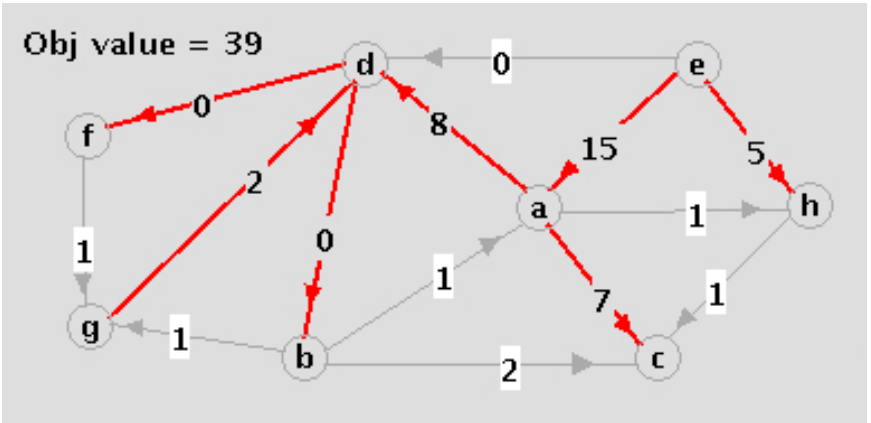
Use dual network simplex method.
Leaving arc: (b,a) Entering arc: (a,d)

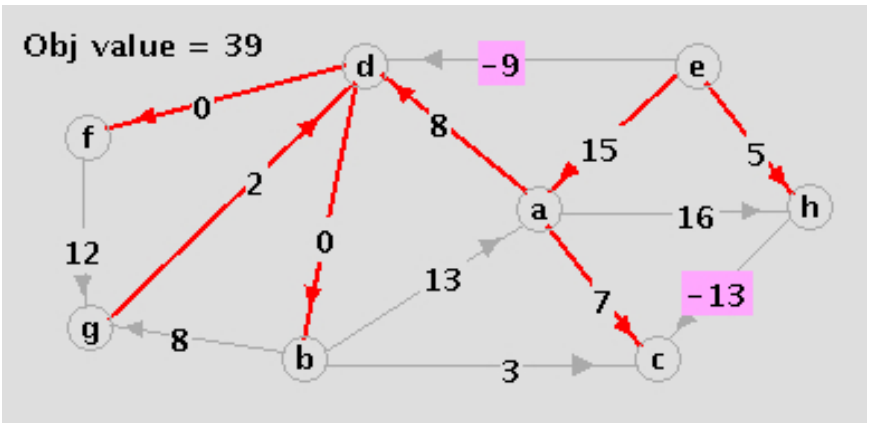


Primal Feasible!

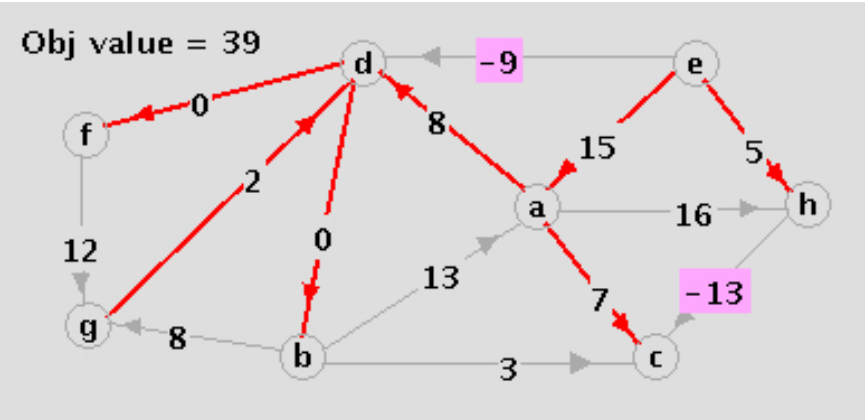
Two-Phase Method–Phase II

- Turn off display of artificial dual slacks.
- Turn on display of dual slacks.

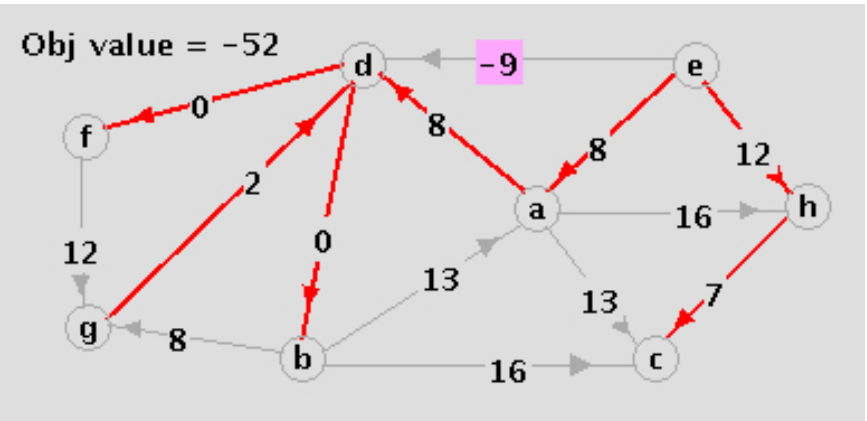




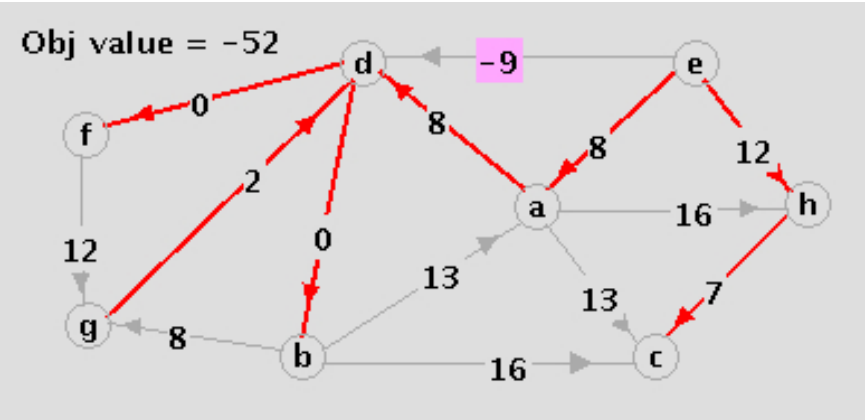
Two-Phase Method–Second Pivot



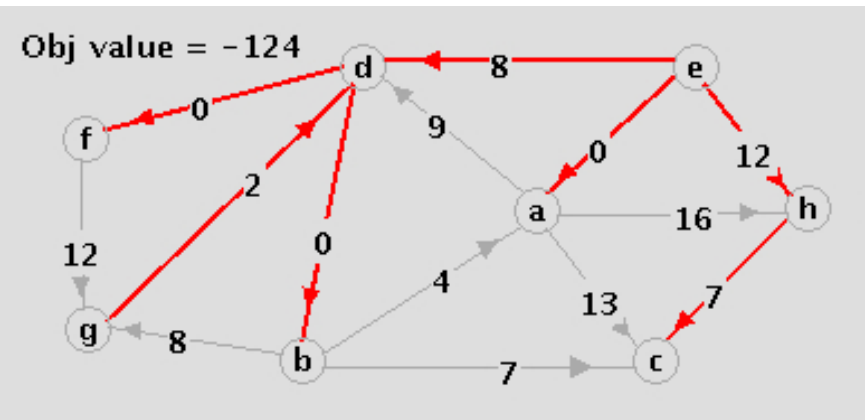
Entering arc: (h,c)
Leaving arc: (a,c)



Two-Phase Method–Third Pivot



Entering arc: (e,d)
Leaving arc: (d,a)



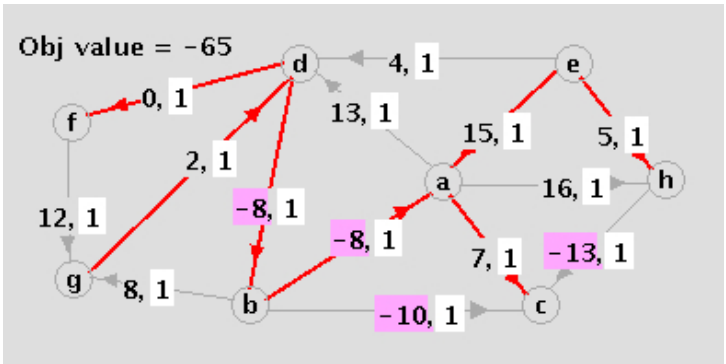
Optimal!

Online Network Simplex Pivot Tool

Click here (or on any displayed network) to try out the online network simplex pivot tool.

One-Phase Primal-Dual Method

- Artificial flows and slacks are multiplied by a parameter μ.
- In the Figure, $_{4,1}$ represents $4 + 1\mu$.
- *Question:* For which μ values is dictionary optimal?
- *Answer:*

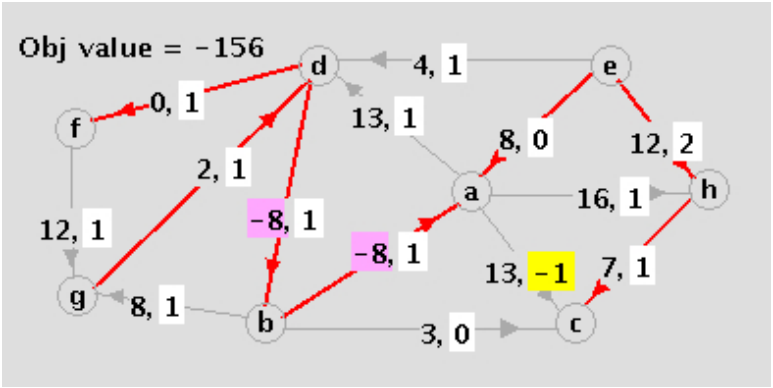


$$
\begin{array}{rcccll}
7 & + & \mu & \geq & 0 & (a,c) \\
13 & + & \mu & \geq & 0 & (a,d) \\
16 & + & \mu & \geq & 0 & (a,h) \\
-8 & + & \mu & \geq & 0 & (b,a) \\
-10 & + & \mu & \geq & 0 & (b,c) \\
8 & + & \mu & \geq & 0 & (b,g) \\
-8 & + & \mu & \geq & 0 & (d,b)
\end{array}
\qquad
\begin{array}{rcccll}
 & & \mu & \geq & 0 & (d,f) \\
15 & + & \mu & \geq & 0 & (e,a) \\
4 & + & \mu & \geq & 0 & (e,d) \\
5 & + & \mu & \geq & 0 & (e,h) \\
12 & + & \mu & \geq & 0 & (f,g) \\
2 & + & \mu & \geq & 0 & (g,d) \\
-13 & + & \mu & \geq & 0 & (h,c)
\end{array}
$$

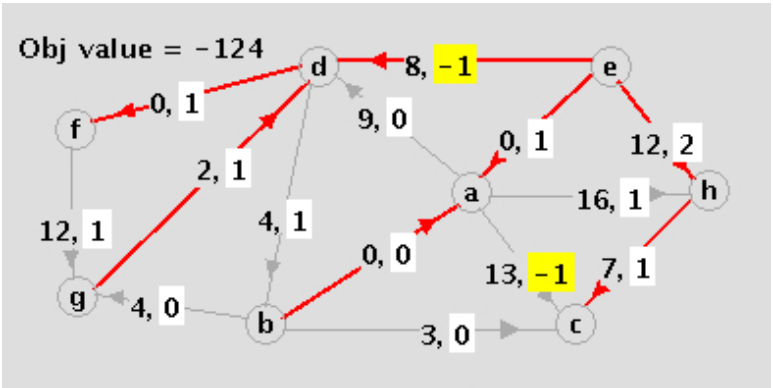
- That is, $13 \leq \mu < \infty$.
- Lower bound on μ is generated by arc (h,c).
- Therefore, (h,c) enters.
- Arc (a,c) leaves.

Second Iteration

- Range of μ values: $8 \leq \mu \leq 13$.
- Leaving arc: (d,b)
- Entering arc: (e,d)



New tree is OPTIMAL!



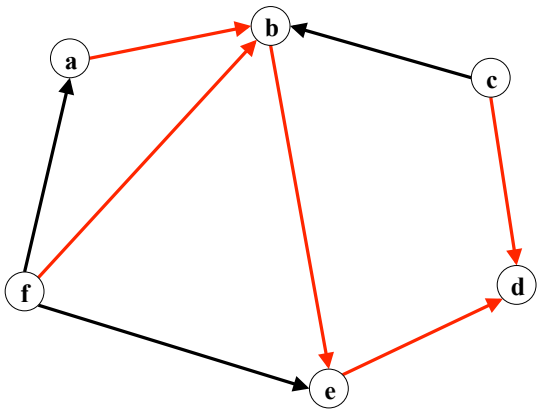
Online Network Simplex Pivot Tool

Click here (or on any displayed network) to try out the online network simplex pivot tool.

Planar Networks (NOTE: Not Covered)

Definition. *Network is called planar if can be drawn on a plane without intersecting arcs.*

Theorem. *Every planar network has a geometric dual—dual nodes are faces of primal network.*



Note:

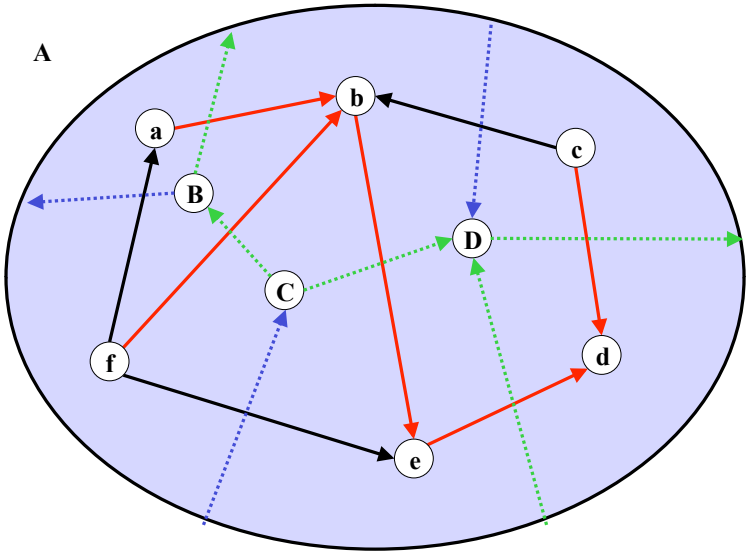
- Primal spanning tree shown in red.

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*

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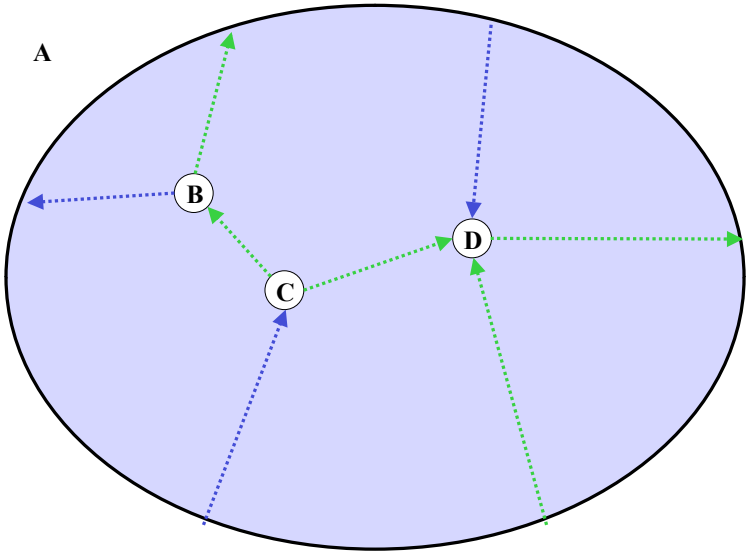
- Dual node A is "node at infinity".
- Primal spanning tree shown in red.
- Dual spanning tree shown in blue (don't forget node A).

Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*

Planar Networks (NOTE: Not Covered)

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Notes:

- Dual node A is "node at infinity".
- Primal spanning tree shown in red.
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Theorem. *A dual pivot on the primal network is exactly a primal pivot on the dual network.*

Integrality Theorem

Theorem. *Assuming integer supplies, every basic (i.e. tree) solution assigns integer flow to every arc.*

Corollary. *Assuming integer supplies, every basic optimal solution assigns integer flow to every arc.*