MSC COURSE IN MATHEMATICS AND FINANCE

Advanced Topics in Volatility Modelling
Final Exam, May 2015

Answer all questions. The total number of points is 100*, and the precise grading is indicated in the text. The rigour and clarity of your answers will be taken into account in the final grade.*

We shall consider a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as given a priori; all the processes are assumed to lie on this probability space and are adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Each problem is independent of the others.

1. [15 points] Warm-up: Preliminary questions

(i) [2 points] Recall the definition of a continuous $\mathcal{F}$ martingale.

(ii) [2 points] Recall the definition of a continuous $\mathcal{F}$-local martingale, and of a strict local martingale.

(iii) [2 points] Let S be the unique strong solution to the stochastic differential equation $\mathrm{d}S_t = \sigma S_t \mathrm{d}W_t$, $S_0 = s > 0$, $\sigma > 0$, where W is an $\mathcal{F}$-adapted standard Brownian motion. Compute the implied volatility in this model, for each maturity $T \geq 0$ and strike $K \geq 0$.

(iv) [3 points] Define and explain in no more than five lines what the local volatility is.

(v) [6 points] Give an example (without proof) of a strict local martingale. Let now S denote a strictly positive strict local martingale, and assume no interest rate nor dividend. Fix a maturity $T > 0$ and consider the two functions $f, g : [0, \infty) \to \mathbb{R}$ defined by:

$$f : K \mapsto \mathbb{E}\left(S_T - K\right)_+,$$
$$g : K \mapsto \mathbb{E}\left(K - S_T\right)_+ + S_0 - K.$$

Explain any arbitrage issues using either (or both) function as a definition of a European Call option at inception. What are the consequences on the implied volatility?

2. [25 points] Problem: the SVI parameterisation

We consider here a stock price process $(S_t)_{t\geq 0}$, assumed to be a true martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $S_0 = 1$. The strictly positive real number $K > 0$ shall always represent the strike of a European option, and $x := \log(K)$ the log strike (or log-moneyness). The goal of this problem is to come up with a robust calibration procedure for the implied volatility. The parameterisation to be calibrated is the so-called 'Stochastic Volatility Inspired' (SVI) parameterisation, where the total implied variance is assumed to have the following representation, for each maturity:

$$\sigma^2_{\text{SVI}}(x) = a + b\left\{\rho(x-m) + \sqrt{(x-m)^2 + \xi^2}\right\}, \tag{2.1}$$

where $a, \xi \geq 0$, $b, m \in \mathbb{R}$, $\rho \in [-1, 1]$. We shall call it here the implied volatility smile, with a slight abuse of language. Calibration on, say the S&P500 data, means finding the optimal parameters a, b, ρ, m, ξ, that minimises the (L^2) distance between SVI and the observed data.

(i) [5 points] Compute the following quantities:

$$\sigma^2_{\text{SVI}}(0), \quad \partial_x \sigma^2_{\text{SVI}}(0), \quad \partial_{xx}\sigma^2_{\text{SVI}}(0), \quad \lim_{x\uparrow\infty} \frac{\sigma^2_{\text{SVI}}(x)}{x}, \quad \lim_{x\downarrow-\infty} \frac{\sigma^2_{\text{SVI}}(x)}{|x|}, \quad \min_{x\in\mathbb{R}} \sigma^2_{\text{SVI}}(x), \quad \operatorname{argmin} \sigma^2_{\text{SVI}}(x).$$

(ii) [2 points] Assume that the observed smile is symmetric around the at-the-money (as is often the case on FX markets). Find conditions on the SVI parameters to ensure this.

(iii) [2 points] Assume that the observed smile is strictly decreasing (as is often the case on Equity markets for small maturities). Find conditions on the SVI parameters ensuring this.

(iv) [3 points] Using Roger Lee's moment formula (Theorem A.1 on page 5), determine necessary conditions on the parameters in order to avoid arbitrage.

(v) [8 points] We assume that the data allows us to consider $m = 0$. From the data, we observe the at-the-money implied volatility w_0, the small-and large-strike slopes of the smile, as well as the at-the-money skew w_1 and curvature w_2. Provide an algorithm to calibrate SVI perfectly to these observed values. From this analysis, is the assumption $m = 0$ realistic?

(vi) [5 points] From now on, we denote by $\sigma_{\text{imp}}(x, t)$ the implied volatility (not total implied variance) at the logmoneyness level $x \in \mathbb{R}$ and for the maturity t. Assume now that your boss has just hired a young quantitative researcher (not from Imperial College!!!), who suggests to use the following parameterisation for the implied volatility smile:

$$\sigma_{\text{imp}}(x, t) = |x - m|\mathrm{e}^{-\beta t}, \qquad \text{for all } x \in \mathbb{R}, t \geq 0$$

where $m \in \mathbb{R}$ and $\beta > 0$ are two parameters giving flexibility to the model. Using Roger Lee's Moment Formula (Theorem A.1 on page 5) and no-arbitrage properties in time, explain to your boss why this model should not be used.

3. [30 points] Problem: Tail asymptotics of the implied volatility

(i) [3 points] Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Recall the definition of $\limsup_{n\uparrow+\infty} u_n$ and $\liminf_{n\uparrow+\infty} u_n$. Give a simple example of a sequence $(u_n)_{n\in\mathbb{N}}$ for which $\limsup_{n\uparrow+\infty} u_n \neq \liminf_{n\uparrow+\infty} u_n$.

(ii) [3 points] Study the smoothness of the function ψ defined in (A.1) in Appendix A, and its behaviour at zero and infinity.

(iii) [5 points] Consider the Black-Scholes model, in which the stock price process is the unique strong solution to the stochastic differential equation $\mathrm{d}S_t = \xi S_t \mathrm{d}W_t$, with $S_0 = 1$, $\xi > 0$, and where W is a standard Brownian motion. In this model, compute the quantities p^*, β_R, q^* and β_L defined in Roger Lee's moment formula (Theorem A.1 in Appendix A). What can you conclude?

(iv) We now consider a more sophisticated model for the stock price: $S_t = S_0 \exp(\alpha t + \sigma W_t + X_t)$, for $t \geq 0$, where W is a standard Brownian motion, $\sigma > 0$, $\alpha \in \mathbb{R}$, and where X is a Variance-Gamma process (a particular example of a pure jump Lévy process), independent of W, and satisfies

$$\log \mathbb{E}\left(\mathrm{e}^{\mathtt{i}uX_t}\right) = -\frac{t}{\nu}\log\left(1 - \mathtt{i}u\theta\nu + \frac{\xi^2\nu u^2}{2}\right),$$

for all $t \geq 0$, $u \in \mathbb{R}$, where ν and θ are strictly positive parameters.

(a) [2 points] For any time $t \geq 0$, compute the characteristic function of the stock price S_t, defined by $\Phi(u) := \mathbb{E}\left(S_t^{\mathtt{i}u}\right)$ for any $u \in \mathbb{R}$.

(b) [2 points] Determine the value of α ensuring that the process $(S_t)_{t\geq 0}$ is a true martingale.

(c) [6 points] For any $t \geq 0$, compute the strip of regularity $\mathcal{D}$ of Φ, defined by

$$\mathcal{D} := \{z \in \mathbb{C} : |\Phi(z)| < \infty\}.$$

What are the conditions on the parameters ensuring that the process S is a true martingale? Deduce p^* and q^* as given in Roger Lee's Wing Formula (Theorem A.1).

(d) [3 points] How do the wings of the implied volatility evolve with maturity? Quote a model (without proof) in which this behaviour is different.

(e) [3 points] Compute the limits of β_L and β_R as ν tends to zero and infinity. What does that imply for the shape of the implied volatility smile.

(f) [3 points] Study the influence of the parameter ξ on the left wing of the smile.

4. [30 points] Problem: Symmetric smiles

We consider here the space $\mathcal{S}$ of strictly positive random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with unit expectation:

$$\mathcal{S} := \{S \text{ on } (\Omega, \mathcal{F}, \mathbb{P}) : S > 0 \text{ almost surely and } \mathbb{E}(S) = 1\}.$$

For any random variable $S \in \mathcal{S}$, we define its symmetrised version $\widehat{S}$ via

$$\mathbb{P}\left(\widehat{S} \leq z\right) = \mathbb{E}\left(S \mathbb{1}_{\{S \geq 1/z\}}\right), \qquad \text{for any } z > 0.$$

(i) [8 points] Prove that if $S \in \mathcal{S}$ then $\mathbb{E}(S^u) = \mathbb{E}\left(\widehat{S}^{1-u}\right)$ for all $u \in [0,1]$. Deduce that $\widehat{S} \in \mathcal{S}$.

(ii) [6 points] Let now $(S_t)_{t \geq 0}$ represent a stock price process such that $S_t \in \mathcal{S}$ for all $t \geq 0$, and $S_0 = 1$. We further assume for simplicity that $S_0 = 1$. Let $x := \log(K)$, $K > 0$, denote the log-moneyness, assume that interest rates are null, and denote by $C_{\mathrm{BS}}(x, t, \sigma)$ the price of a European Call option in the Black-Scholes model with instantaneous volatility σ, maturity equal to t and strike equal to e^x. Prove that for any $t > 0$ and any $\sigma > 0$, the following equalities hold for all $x \in \mathbb{R}$ if $S \in \mathcal{S}$:

$$C_{\mathrm{BS}}(-x, t, \sigma) = 1 - \mathrm{e}^{-x} + \mathrm{e}^{-x} C_{\mathrm{BS}}(x, t, \sigma),$$
$$1 - \mathrm{e}^{-x} + \mathrm{e}^{-x} \mathbb{E}(S - \mathrm{e}^x)_+ = \mathbb{E}\left[S(S^{-1} - \mathrm{e}^{-x})_+\right].$$

(iii) [8 points] Deduce that if $S_t \in \mathcal{S}$ for all $t \geq 0$, then the implied volatility $\sigma_t(x)$ (corresponding to Call options on S) satisfies

$$\sigma_t(-x) = \widehat{\sigma}_t(x),$$

for all $x \in \mathbb{R}$, where $\widehat{\sigma}_t(x)$ represents the implied volatility under $\widehat{S}$.

(iv) [8 points] Deduce from the above that if $(S_t)_{t \geq 0} \in \mathcal{S}$, then the implied volatility smile is symmetric ($\sigma_t(x) = \sigma_t(-x)$ for all $(x, t) \in \mathbb{R} \times (0, +\infty)$) if and only if $\mathbb{E}(S^u) = \mathbb{E}(S^{1-u})$ for all $u \in [0,1]$. You may use without proof the fact that given a moment generating function $M(u) := \int_{\mathbb{R}} \mathrm{e}^{ux} \mu(\mathrm{d}x)$, the finite measure μ can be recovered uniquely if M is finite on some non-empty open interval (a, b) of the real line.

Appendix A. Roger Lee's Moment Formula

Let S be a true non-negative martingale on $(\Omega, \mathcal{F}, \mathbb{P})$. We let $x := \log(K/S_0)$, for $K > 0$, denote the log-moneyness and we assume interest rates are null. Furthermore, for any $(x,t) \in \mathbb{R} \times (0,\infty)$, $\sigma_t(x)$ shall represent the implied volatility corresponding to European option prices with maturity t and log-moneyness x. We recall the following theorem, due to Roger Lee (2004):

Theorem A.1. *Fix some time $t \geq 0$, and define the function $\psi : \mathbb{R}_+ \to \mathbb{R}$ by*

$$\psi(z) \equiv 2 - 4\left(\sqrt{z(1+z)} - z\right). \tag{A.1}$$

Let $p^* := \sup\{p \geq 0 : \mathbb{E}(S_t^{1+p}) < \infty\}$ and $\beta_R := \limsup_{x\uparrow+\infty} \frac{\sigma_t^2(x)t}{x}$. Then

$$p^* = \frac{1}{2\beta_R} + \frac{\beta_R}{8} - \frac{1}{2}, \qquad \text{or equivalently} \qquad \beta_R = \psi(p^*).$$

Similarly, let $q^* := \sup\{q \geq 0 : \mathbb{E}(S_t^{-q}) < \infty\}$ and $\beta_L := \limsup_{x\downarrow-\infty} \frac{\sigma_t^2(x)t}{|x|}$. Then

$$q^* = \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2}, \qquad \text{or equivalently} \qquad \beta_L = \psi(q^*).$$