Lectures 25-26: Futures and Forward Contracts: Valuation.

I. Reading.
   A. BKM, Chapter 22, Sections 22.4.
   B. BKM, Chapter 23, omit Section 23.5.

II. Futures Prices.
   A. Applicability of Spot Forward Parity.
      1. In general, the futures price need not equal the forward price and so spot forward parity need not hold exactly for futures contracts.
      2. However, spot forward parity can be expected to hold approximately for futures.

III. Forward Prices: Spot Forward Parity.
   A. Introduction.
      1. Interested in determining how the forward price is determined.
      2. Turns out that no arbitrage implies that the forward price must be related to the spot price in a very particular way.
      3. Example. WSJ 4/28/97.

Underlying	Spot	Forward - deliver on 12/97	Growth Rate
Gold-1oz	339.9	349.6	2.9%
Cotton-100 lb	70.53	75.96	7.7%
S&P 500	775.2	791.55	2.1%

Why the differences?

B. The Law of One Price: No Carrying Costs.

1. Example 1:

a. Consider a forward contract to deliver 1 oz of gold on 4/98 entered into on the 4/97. The spot price for 1 oz of gold is $400. The price of a U.S. T-bill ($100 face value) maturing on 4/98 is 80.

b. What is the relation between the forward price and the spot price?

c. Can replicate a forward contract by going long the stock and shorting discount bonds that mature on the settlement date with a face value equal to the forward price.

Strategy	4/97	4/98
Buy a forward contract on 4/97 which delivers 1 oz of gold on 4/98	0	$S(4/98) - F_1(4/97)$

Strategy	4/97	4/98
Buy 1 oz of gold on 4/97 and sell on 4/98	-400	S(4/98)
Sell 1-yr U.S. T-bills on 4/97 with face value of $F_1(4/97)$	$F_1(4/97)$ 0.8	$-F_1(4/97)$
Net Cash Flow	$F_1(4/97)$ 0.8 -400	$S(4/98) - F_1(4/97)$

d. Can see that these two strategies have the same cash flows.

e. By definition, no money changes hands today under the forward contract; so the law of one price says

$F_1(4/97)$ 0.80 - 400 = 0; and so
$F_1(4/97) = 400 / 0.8 = 500$.

f. Note:

(1) if $F_1(4/97)$ 0.80 - 400 > 0 (i.e., $F_1(4/97) > 500$), the buyer of the forward contract must be paid money today to be induced to enter the contract.

(2) if $F_1(4/97)$ 0.80 - 400 < 0 (i.e., $F_1(4/97) < 500$), the buyer of the forward contract would pay money today to be allowed to enter the contract.

2. Thus, have shown that the forward price is just the future value of the spot price at the settlement date (invested today in a discount bond maturing at the settlement date):

$F_T(0) = S(0) / d_T(0)$ or $F_T(0) = S(0) [1 + y^*_T(0)]^T$

where

a. T is time to settlement.
b. S(0) be the value of the underlying at time 0.
c. $d_T(0)$ be the price of a T period discount bond with a \$1 face value.
d. $y^*_T(0)$ be the effective 1-period yield on a T period discount bond.
e. $F_T(0)$ be the time 0 forward price of the underlying for delivery in T periods.

3. So at any point in time expect the forward price of gold to increase as the settlement date becomes more distant since the future value of the spot price increases with maturity.

C. General Case: Carrying Costs.

1. Suppose there are certain costs associated with holding or carrying an asset between time 0 and the settlement of the forward contract at time T.
   a. Example: When a physical commodity is the underlying, any storage costs are a carrying cost.
   b. Example: If the underlying is a stock index, dividend payments by the stocks in the index are a negative carrying cost.
2. Assume the carrying costs are known at time 0.
3. Example 2:
   a. Consider a forward contract to deliver 100 lb of cotton on 4/98 entered into on the 4/97. The spot price on 4/97 for 100 lb of cotton is \$71. The price of a U.S. T-bill (\$100 face value) maturing on 4/98 is 80. The cost of storing 100 lb of cotton from 4/97 to 4/98 is \$10 payable on 10/97. The price of a U.S. T-bill (\$100 face value) maturing on 10/97 is 90.
   b. What is the relation between the forward price and the spot price?
   c. Consider the following two investment strategies:

Strategy	4/97	10/97	4/98
Buy a forward contract on 4/97 which delivers 100 lb cotton on 4/98	0	0	$S(4/98)-F_1(4/97)$

Strategy	4/97	10/97	4/98
Buy 100 lb cotton on 4/97 and sell on 4/98	-71	-10	S(4/98)
Buy a ½-year discount bond on 4/97 with face value of 10 and close out at maturity	-10 x 0.9	10	
Sell a 1 year discount bond on 4/97 with face value of $F_1(4/97)$ and hold to maturity	$F_1(4/97)$ 0.8		$-F_1(4/97)$
Net Cash Flow	$F_1(4/97)$ 0.8 - 10x 0.9 - 71	0	$S(4/98)-F_1(4/97)$

d. Can see that these two strategies have the same cash flows.
e. By definition, no money changes hands today under the forward contract; so the law of one price says:

$0 = F_1(4/97)\ 0.8 - 10 \times 0.9 - 71$

f. So

$F_1(4/97) = (71 + 9)/0.8 = 100.$

4. Thus, get a relation between the current spot price and the forward price:

$F_T(0)\ d_T(0) - C(t_c)\ d_{tc}(0) = S(0)$

or

$$F_T(0)\ \frac{1}{[1+y^*_T(0)]^T} - C(t_c)\ \frac{1}{[1+y^*_{tc}(0)]^{tc}} = S(0)$$

where:

a. the carrying costs $C(t_c)$ are paid at time t_c between times 0 and T (usually will take the settlement date T to be the date at which the costs are paid).
b. S(0) be the value of the underlying at time 0.
c. $d_\tau(0)$ be the price of a τ period discount bond with a \$1 face value.
d. $y^*_\tau(0)$ be the effective 1-period yield on a τ period discount bond.
e. $F_T(0)$ be the time 0 forward price of the underlying for delivery in T periods.

5. This relation is known as spot forward parity and can be rewritten:

$$F_T(0) = [1+y^*_T(0)]^T\ [\ S(0) + C(t_c)\ \frac{1}{[1+y^*_{tc}(0)]^{tc}}]$$

6. Can see that
   a. a positive carrying cost implies a higher forward price.
   b. a negative carrying cost (e.g., dividend-paying underlying) implies a lower forward price.
7. Example 3:
   a. The S&P 500 index is 800 on 4/97. The price of a discount bond (face vaue of 100) maturing on 4/98 is 80. The stocks in the index will pay dividends amounting to 40 of index value on 10/97. The price of a discount bond (face vaue of 100) maturing on 10/97 is 90. What is forward price on 4/97 for delivery of the index on 4/98?
   b. Use spot-forward parity

$F_1(4/97)\ 0.8 - (-40)\ 0.9 = 800 \Rightarrow F_1(4/97) = (800 - 36)/0.8 = 955.$

D. Application to Foreign Currency Forward Contracts: Covered Interest Parity.

1. In the case of foreign currency forward contracts, spot forward parity is known as covered interest parity.
2. For a foward contract to deliver a foreign currency in T years, the underlying is the foreign currency. The negative carrying cost is the interest received from investing the foreign currency in a discount bond maturing in T years.
3. Example 4:
   a. The spot price for a British pound on 4/97 is \$1.60: i.e., $S^{\$/£}(4/97) = 1.60$. The yield on a 1 year discount bond denominated in U.S. dollars is 25% while the yield on a 1 year discount bond denominated in pounds is 10%. What is the forward price on 4/97 for delivery of one £ on 4/98?
   b. Consider the following two strategies:

Strategy	4/97	4/98
Buy \$1 of 1-year \$-denominated discount bonds on 4/97 and sell on 4/98	-1	1 + 0.25

Strategy	Time 0	Time T
Sell a forward contract which delivers [1/1.6][1+ 0.1] = 0.6875 £ on 4/98	0	$\{F_1^{\$/£}(4/97)-S^{\$/£}(4/98)\}$ x [1/1.6][1+0.1] = $F_1^{\$/£}(4/97)$ [1/1.6][1+0.1] - $S^{\$/£}(4/98)\}$ x [1/1.6][1+0.1]
Buy \$1 worth of £ on 4/97 (£1/1.6) and invest in 1-year £-denominated discount bonds and hold til maturity. On 4/98, convert the [1/1.6][1+ 0.1] = 0.6875 £ to \$ at $S^{\$/£}(4/98)$.	-1	$S^{\$/£}(4/98)$[1/1.6][1+0.1]
Net Cash Flow		$F_1^{\$/£}(4/97)$ [1/1.6][1+0.1]

   c. The first strategy is buying a 1-year \$-denominated discount bond while the second is creating a synthetic 1-year \$-denominated discount bond on 4/97 by:
      (1) buying pounds on 4/97,
      (2) investing the proceeds in 1-year £-denominated discount bonds, and
      (3) locking in on 4/97 the exchange rate (the forward rate) at which the pounds can be converted back to dollars on 4/98.

d. Can see that these two strategies cost \$1 on 4/97 and generate a certain dollar cash flow on 4/98. The law of one price says that the certain dollar cash flows on 4/98 must be the same:

$1 + 0.25 = F_1^{\$/£}(4/97)\ [1/1.6][1+0.1]$

and so $F_1^{\$/£}(4/97) = \$1.8181/£$.

4. Thus obtain the following result which is the covered interest parity theorem:

$[1+y^{*\$}_T(0)]^T = [1+y^{*£}_T(0)]^T\ F_T^{\$/£}(0)/S^{\$/£}(0)$

where

a. $y^{*\$}_T(0)$ is the effective per period yield on a T period discount bond denominated in U.S. dollars.
b. $y^{*£}_T(0)$ is the effective per period yield on a T period discount bond denominated in £.
c. $S^{\$/£}(0)$ is the spot price of 1 £ at time 0.
d. $F_T^{\$/£}(0)$ be the forward price at time 0 for 1 £ delivered in T periods.

5. Can see that:
   a. if the yield on the foreign currency discount bond is lower than on the dollar-denominated discount bond, the forward price of the foreign currency (in \$s) is higher that the spot price (in \$s).
   b. if the yield on the foreign currency discount bond is higher than on the dollar-denominated discount bond, the forward price of the foreign currency (in \$s) is lower that the spot price (in \$s).

E. Spot Forward Parity and Arbitrage.

1. If spot forward parity is violated, there is an arbitrage opportunity.
2. Example 1 (cont):
   a. Suppose the forward price on 4/97 for delivery of 1 oz of gold on 4/98 is 520 which is greater than the 500 implied by spot forward parity.
   b. So the forward price is too high which implies that you want to sell forward contracts and buy the underlying:

Strategy	4/97	4/98
Sell a forward contract on 4/97 which delivers 1 oz of gold on 4/98	0	520 - S(4/98)
Buy 1 oz of gold on 4/97 and sell on 4/98	-400	S(4/98)
Sell 1-yr U.S. T-bills on 4/97 with face value of 520	520 x 0.8	-520
Net Cash Flow	16	0

   c. This strategy is an arbitrage opportunity.
3. Example 4 (cont):
   a. Suppose the forward price on 4/97 for delivery of a £ on 4/98 is $2 which is higher than $1.8181 implied by covered interest parity.
   b. Want to buy the synthetic 1-year $-denominated discount bond and sell the 1-year $-denominated discount bond:

Strategy	4/97	4/98
Sell $1 of 1-year $-denominated discount bonds on 4/97 and close out on 4/98	1	-1.25
Sell a forward contract which delivers [1/1.6][1+ 0.1] = 0.6875 £ on 4/98	0	$\{2-S^{\$/£}(4/98)\}$ x 0.6875 $=2 \times 0.6875 - S^{\$/£}(4/98)\ 0.6875$
Buy $1 worth of £ on 4/97 (£1/1.6) and invest in 1-year £-denominated discount bonds and hold til maturity. On 4/28, convert the [1/1.6][1+ 0.1] = 0.6875 £ to $ at $S^{\$/£}(4/98)$.	-1	$S^{\$/£}(4/98)\ 0.6875$
Net Cash Flow	0	2 x 0.6875 - 1.25 = 0.125

   c. This strategy is an arbitrage opportunity.