Lectures 17-19: Bond Pricing, Forward Rates and the Yield Curve.

I. Reading.
   A. BKM, Chapter 15.

II. Discount Factors, Bond Prices and No Arbitrage.
   A. Discount Factors.
      1. The τ-period discount factor at time t, denoted $d_\tau(t)$, is the price at t of \$1 received for certain at time $t+\tau$.
      2. The τ-period discount factor at time t is analogous to the PVIF discussed in the time value of money.
      3. If the price of a discount bond paying $C_\tau(t+\tau)$ in τ periods is $p_\tau(t)$, then $d_\tau(t)$ is given by $p_\tau(t)/C_\tau(t+\tau)$.
      4. In these notes, a period is equal to a year unless otherwise stated.
      5. Example:
         a. Government note and strip prices for 2/15/95.
         b. One period is a year.

Government Bonds and Notes.

Rate	Maturity	Ask Price
4	Aug 95	?
4	Feb 96	?
6	Aug 96	?

U.S. Treasury Strips.

Type	Maturity	Ask Price
ci	Aug 95	97
ci	Feb 96	94
ci	Aug 96	90

         c. Can calculate $d_{½}$(Feb 95) as follows using the Aug 95 strip: $d_{½}(\text{Feb 95}) = p_{½}(\text{Feb 95})/100 = 97/100 = 0.97$.
         d. Can calculate d_1(Feb 95) as follows using the Feb 96 strip: $d_1(\text{Feb 95}) = p_1(\text{Feb 95})/100 = 94/100 = 0.94$.
         e. Can calculate $d_{1½}$(Feb 95) as follows using the Aug 96 strip: $d_{1½}(\text{Feb 95}) = p_{1½}(\text{Feb 95})/100 = 90/100 = 0.90$.

B. An Arbitrage Opportunity.

1. Definition: An investment that does not require *any* cash outflows and generates a strictly positive cash inflow with some probability is known as an arbitrage opportunity.
2. In well functioning markets arbitrage opportunities can not exist since any individual who prefers more to less wants to invest as much as possible in the arbitrage opportunity.

C. No Arbitrage and the Law of One Price.

1. The absence of arbitrage implies that any two assets with the same stream of riskless cash flows must have the same price.
2. This implication is known as the law of one price.
3. Otherwise, could buy the lower priced asset and sell the higher priced asset and earn an arbitrage profit.
   a. Zero cash flows in the future.
   b. Positive cash flow today.
4. Example: WSJ for 2/18/97. Compare U.S. Treasury Strips maturing on the same date. The differences in price are very small.

D. Fixed Income Prices and the Law of One Price.

1. Any bond i paying the certain cash flow stream, $C^i(t+½)$, $C^i(t+1)$,..., $C^i(t+N)$, must have the following price at time t for there to be no arbitrage:

$$P^i(t) = d_{½}(t)\, C^i(t+½) + d_1(t)\, C^i(t+1) + \ldots + d_N(t)\, C^i(t+N).$$

2. Example (cont): Since above we calculated the discount factors at Feb 95 for Aug 95, Feb 96 and Aug 96, we can determine the value each of the three bonds/ notes in the absence of arbitrage using this formula:

a. 4 Aug 95:

$$P^{4\text{Aug }95}(\text{Feb95}) = d_{½}(\text{Feb 95}) \times [100 + 4/2] = 0.97 \times 102 = 98.94.$$

b. 4 Feb 96:

$$P^{4\text{Feb }96}(\text{Feb95}) = d_{½}(\text{Feb95}) \times [4/2] + d_1(\text{Feb95}) \times [100 + 4/2]$$
$$= 0.97 \times 2 + 0.94 \times 102 = 97.82.$$

c. 6 Aug 96:

$$P^{6\text{Aug}96}(\text{Feb95}) = d_{½}(\text{Feb95}) \times [6/2] + d_1(\text{Feb95}) \times [6/2] + d_{1½}(\text{Feb95}) \times [100 + 6/2]$$
$$= 0.97 \times 3 + 0.94 \times 3 + 0.90 \times 103 = 98.43.$$

E. How to Earn an Arbitrage Profit when the Law of One Price is Violated.

1. Basic Idea.
   a. Buy the undervalued assets and sell the overvalued assets.
   b. Need to choose the weights to ensure that *all* cash flows are zero or positive.
   c. Easiest way is to choose the weights so that all future cash flows are zero; then today's cash flow will be positive if there is an arbitrage opportunity.
2. How to use coupon bonds and a mispriced strip to create an arbitrage position.
   a. Example (cont): Suppose the 4 Feb 96 note is priced at 98 on 2/15/95 which is too high relative to the prices of the Aug 95 and Feb 96 strips.
      (1) Must sell the 4 Feb 96 note.
      (2) The idea is to
         (a) sell the 4 Feb 96 note (overpriced); and
         (b) buy a "synthetic" 4 Feb96 note created using the Aug 95 and Feb 96 strips.
      (3) Must buy the Aug 95 and Feb 96 strips.
      (4) More specifically:
         (a) let *a* be the number of Feb 96 strips bought.
         (b) let *b* be the number of Aug 95 strips bought.

Position		2/15/95	8/15/95	2/15/96
Sell 1 4 Feb 96 note	SELL NOTE	1 x 98 = 98	-1 x 4/2 = -2	-1 x {100+4/2} = -102
Buy *a* Feb 96 strips	BUY SYNTHETIC NOTE	-*a* x 94		*a* 100 = 102
Buy *b* Aug 95 strips		-*b* x 97	*b* 100=2	
Net		98 - *a* 94 - *b* 97	0	0

      (5) So *a* 100 = 102 implies *a* = 1.02.
      (6) So *b* 100 = 2 implies *b* = 0.02.
      (7) His position earns an arbitrage profit of:

98 - *1.02* x 94 - *0.02* x 97 = 0.18 today.

      (8) It seems like a lot of trouble for 18 cents. But sell 1M of the 4 Feb 96 note and the profit becomes $180000.

III. Yields, Yield Curves, Prices and Discount Factors.

A. Relation between Prices and Yields for Discount Bonds.

1. Yields are usually quoted in the industry as APRs with semiannual compounding; ie as bond equivalent yields.
2. As above, let $p_\tau(t)$ be the price at time t on a τ-year discount bond with face value $C_\tau(t+\tau)$.
3. For discount bonds, yield (expressed as an APR with semiannual compounding) is related to price in the following way:

$$y_\tau(t) = 2 \left\{ \left[\frac{C_\tau(t+\tau)}{p_\tau(t)} \right]^{1/(2\tau)} - 1 \right\} \Leftrightarrow p_\tau(t) = \frac{C_\tau(t+\tau)}{\left[1 + \frac{y_\tau(t)}{2}\right]^{2\tau}} .$$

4. Example (cont):
   a. Can calculate the yield on a six month discount bond (expressed as an APR with semi-annual compounding) using the price of the Aug 95 strip:

$y_{½}$ (Feb 95) = 2 x {100/97 -1} = 6.186%.

   b. Can calculate the yield on a one year discount bond (expressed as an APR with semi-annual compounding) using the price of the Feb 96 strip:

y_1 (Feb 95) = 2 x $\{[100/94]^{1/2} - 1\}$ = 6.284%.

   c. Can calculate the yield on a 1.5 year discount bond (expressed as an APR with semi-annual compounding) using the price of the Aug 96 strip:

$y_{1½}$ (Feb 95) = 2 x $\{[100/90]^{1/3} - 1\}$ = 7.149%.

5. These formulas are used below to calculate the yield on a discount bond when the relevant discount factor is known and to calculate the relevant discount factor when the discount bond's yield is known.

B. Relation between Discount Factors and Yields for Discount Bonds.

1. Discount bonds yield (expressed as an APR with semiannual compounding) is related to the relevant discount factor by setting price equal to the discount factor and the payoff at maturity equal to 1 in the formula above:

$$y_\tau(t) = 2 \left\{ \left[\frac{1}{d_\tau(t)} \right]^{1/(2\tau)} - 1 \right\} \Leftrightarrow d_\tau(t) = \frac{1}{\left[1 + \frac{y_\tau(t)}{2}\right]^{2\tau}} .$$

2. Example (cont):

a. Can calculate the ½ year discount factor on 2/15/95 using the yield of the Aug 95 strip (expressed as an APR with semi-annual compounding):

$$d_{½}(Feb\ 95) = \frac{1}{\left[1 + \frac{0.06186}{2}\right]^{1}} = 0.97.$$

b. Can calculate the 1 year discount factor on 2/15/95 using the yield of the Feb 96 strip (expressed as an APR with semi-annual compounding):

$$d_{1}(Feb\ 95) = \frac{1}{\left[1 + \frac{0.06284}{2}\right]^{2}} = 0.94.$$

c. Can calculate the 1½ year discount factor on 2/15/95 using the yield of the Aug 96 strip (expressed as an APR with semi-annual compounding):

$$d_{1½}(Feb\ 95) = \frac{1}{\left[1 + \frac{0.07149}{2}\right]^{3}} = 0.90.$$

C. Yield Curve.

1. Discount bonds of differing maturities can have different yields to maturity.
2. The yield curve is just the yield to maturity (YTM) on a n-year discount bond graphed as a function of n.
3. Note that the YTM on the shortest maturity discount bond of interest is known as the spot rate.
4. It is not correct to use the yield to maturity on a n-year coupon bond as the yield on a n-year zero coupon bond: these are not the same.
5. Example: 2/15/95
   a. Yield on a 1½ year discount bond (expressed as an APR with semi-annual compounding) is 7.149%.
   b. YTM on the 6 Aug 96 note is 7.122% since

$$P^{6Aug96}(Feb95) = (6/2) \times PVAF_{(7.122/2)\%,3} + 100 \times PVIF_{(7.122/2)\%,3} = 8.40 + 90.03 = 98.43$$

which is lower than the yield on a 1½ year discount bond.

   c. The reason is that the note has coupon cash flows prior to Aug 96 and the yields on ½ year and 1 year discount bonds are less than 7.149%.
   d. YTM on the 10 Aug 96 note is 7.106% which is even lower than for the 6 Aug 96 note since a greater portion on the 10 Aug 96's cash flows occur prior to Aug 96 than the 6 Aug 96 note.

IV. Calculation of Discount Factors.

A. Introduction

1. Short end can be obtained from U.S. T-bill yields.
2. Long end can be obtained from U.S. Treasury strips.
3. It is also possible to obtain the yield curve using U.S. Treasury notes and bonds despite the presence of coupon payments.
    a. It is not correct to use the yield to maturity on a n-year coupon bond as the yield on a n-year zero coupon bond: these are not the same.
    b. The concept of no arbitrage in well functioning markets can be used.

B. Discount Factors from Coupon Bonds.

1. Generally, to replicate a T-period discount bond using coupon bonds:
    a. Use the T-period coupon bond to generate the time T cash flow; so buy the T-period coupon bond.
    b. Use the (T-1) period bond to offset the (T-1)th coupon from the T-period coupon bond; so sell the (T-1)-period coupon bond.
    c. Use the (T-2)-period bond to offset the sum of time (T-2) coupons for the T-period and (T-1)-period coupon bond.
    d. Continue until the cash flows for all points in time between now and time T are zero.
2. Once a discount bond has been replicated, its price and cashflow can be used to calculate the relevant discount factor.
3. Once discount factors have been obtained, yields on discount bonds can be calculated using the formulas described above.

4. Example (cont): 2/15/95

Government Bonds and Notes.

Rate	Maturity	Ask Price
4	Aug 95	98.94
4	Feb 96	97.82
6	Aug 96	98.43

a. Can recover the six month discount factor at 2/15/95 using the Aug 95 note since it has only one payment left on 8/15/95:

$d_{1/2}$ (Feb 95) = 98.94/(100 + [4/2])= 0.97.

b. Can replicate a 1 year discount bond (with par value=100) at 2/15/95 using the 4 Aug 95 and the 4 Feb 96 notes

(1) More specifically:

(a) let *a* be the number of 4 Feb 96 notes bought.

(b) let *b* be the number of 4 Aug 95 notes bought.

Position		2/15/95	8/15/95	2/15/96
Buy *a* 4 Feb 96 notes	BUY SYNTHETIC DISCOUNT BOND	-*a* x 97.82	*a* 2	*a* 102
Buy *b* 4Aug 95 notes		-*b* x 98.94	*b* 102	
Net		- *a* 97.82 - *b* 98.94	0	100

(2) So *a* 102 = 100 implies *a* = 0.9804.

(3) So *a* 2 + *b* 102 = 0 implies *b* = -0.0192. Since *b* is negative, this means that strategy involves selling 0.0192 of the 4 Aug 95 notes.

(4) Thus, the cost of the synthetic discount bond is

0.9804 x 97.82 - *0.0192* x 98.94 = 94.

(5) So

d_1 (Feb 95) = 94/100 = 0.94.

V. Other Bond Pricing Issues.

A. Introduction.
   1. Sometimes prices appear to violate the law of one price.
   2. Usually there is a reason for an apparent violation.
   3. Many fixed income traders are constantly looking for arbitrage opportunities. They are in a position to act on any such opportunity immediately. Thus, it is unlikely that an arbitrage opportunity could be identified by looking at the WSJ.

B. Reasons for deviations from the law of one price.
   1. Taxes.
      a. If two fixed income instruments have the same pretax cash flows but these cash flows are taxed differently, then the two instruments may have different prices
   2. Liquidity.
      a. If the liquidity of two fixed income instruments differ, so too may their prices.
   3. Transaction Costs.
      a. When there are transaction costs in the form of a bid ask spread, no arbitrage implies the ask price for a given instrument must be greater than some cutoff and its bid price must be less than another cutoff.

C. Default Risk.
   1. Fixed income instruments with the same promised cash flows may have different prices if they have differing probabilities of default.
   2. If a fixed income instrument has some probability of default:
      a. expected cash flows from the instrument are less than the promised cash flows.
      b. there is uncertainty or risk associated with the cash flows; so investors require a higher expected return to hold a corporate compared to an equivalent treasury.
   3. Hence, the yield on a discount bond with default risk (e.g., corporate debt) will be greater than the yield on a discount bond of the same maturity without any default risk (e.g., government bills).

1 Period Strip	Treasury	Corporate	
Promised Payment P	110	110	110
Prob of Default	0	0.5	0.5
Expected Payment E[CF]	110	55	55
Expected Return E[R]	10%	10%	37.5%
Price	100	50	40
YTM	10%	120%	175%

VI. Holding Period Return.

A. Definition.

1. Let $h^i(t+½)$ denote the ½ year holding period return on bond i over the ½ year ending at t+½.
2. The ½ year holding period return is defined as follows:

$$h^i(t) = \{p^i(t+½) + C^i(t+½) - p^i(t)\}/p^i(t)$$

where
$p^i(t)$ is the price of the bond at time t; if the bond matures at time t, $p^i(t)$ is the face value of the bond;
$C^i(t)$ is the coupon payment at time t on the bond.

3. The holding period return on a bond is analogous to the return on an asset (which has been defined already).

4. Example: Government strip, note and bond prices 2/15/95

Government Bonds and Notes.

Rate	Maturity	Ask Price	Ask Yield
6	Aug 96	98.43	7.122%

U.S. Treasury Strips.

Type	Maturity	Ask Price	Ask Yield
ci	Aug 95	97	6.186%
ci	Feb 96	94	6.284%
ci	Aug 96	90	7.149%

Government strip, note and bond prices 8/15/95

Government Bonds and Notes.

Rate	Maturity	Ask Price	Ask Yield
6	Aug 96	97.7	8.446%

U.S. Treasury Strips.

Type	Maturity	Ask Price	Ask Yield
ci	Aug 95	100	-
ci	Feb 96	98	4.082%
ci	Aug 96	92	8.514%

a. What is the 6 month holding period return on Aug 95 strips from 2/15/95 to 8/15/95?

$h^{\text{Aug 95 strip}}(8/15/97) = (100-97)/97 = 3.093\%$

(which is 2 x 3.093% = 6.186% when expressed as an APR with semi-annual compounding).

b. What is the 6 month holding period return on Feb 96 strips from 2/15/95 to 8/15/95?

$h^{\text{Feb 96 strip}}(8/15/97) = (98-94)/94 = 4.255\%$

(which is 2 x 4.255% = 8.510% when expressed as an APR with semi-annual compounding).

c. What is the 6 month holding period return on the 6 Aug 96 notes from 2/15/95 to 8/15/95?

$h^{\text{6 Aug 96 note}}(8/15/97) = (97.7+3-98.43)/98.43 = 2.306\%$

(which is 2 x 2.306% = 4.612% when expressed as an APR with semi-annual compounding).

Instrument	Holding Period Return (2/15/95-8/15/95)	YTM	
		Start (2/15/95)	End (8/15/96)
6% Aug 96 Note	4.612%	7.122%	8.446%
Feb 96 Strip	8.510%	6.284%	4.082%

d. Sensitivity Analysis for the 6% Aug 96 Note.

Instrument	Holding Period Return (2/15/95-8/15/95)	(2/15/95)		(8/15/95)	
		YTM	Price	YTM	Price
6% Aug 96 Note	4.612%	7.122%	98.43	8.446%	97.7
6% Aug 96 Note	7.122%	7.122%	98.43	7.122%	98.94
6% Aug 96 Note	8.317%	7.122%	98.43	6.500%	99.52

B. Note.

1. The ½ year holding period return on a ½ year discount bond equals its yield to maturity.
   a. Example. YTM on Aug 95 strip at 2/15/95 is 6.186% which equals its holding period return (expressed as an APR with 6 month compounding) over the ½ year period 2/15/95 to 8/15/95.

2. For any bond of longer maturity, its ½ year holding period return need not equal either its own yield to maturity or the YTM on a ½ year discount bond.
   a. Example. Holding period return (expressed as an APR with 6 month compounding) of the 6 Aug 96 note is 4.612% over the ½ year period 2/15/95 to 8/15/95. YTM on Aug 95 strip at 2/15/95 is 6.186% while the YTM on the 6 Aug 96 note at 2/15/95 is 7.122%.

3. For any bond, if its YTM at the end of the holding period is greater than its YTM at the start of the holding period, then its holding period return will be less than its YTM at the start of the holding period.
   a. Example. YTM on the 6 Aug 96 note at 8/15/95 is 8.446% which is greater than its YTM at 2/15/95 of 7.122%. Thus, its holding period return (expressed as an APR with 6 month compounding) over the ½ year period 2/15/95 to 8/15/95 is 4.612% which is less than itsYTM as of 2/15/95 of 7.122%.

4. For any bond, if its YTM at the end of the holding period is less than its YTM at the start of the holding period, then its holding period return will be greater than its YTM at the start of the holding period.
   a. Example. YTM on the Feb 96 strip at 8/15/95 is 4.082% which is less than its YTM at 2/15/95 of 6.284%. Thus, its holding period return (expressed as an APR with 6 month compounding) over the ½ year period 2/15/95 to 8/15/95 is 8.51% which is greater than itsYTM as of 2/15/95 of 6.284%.

5. When a bond pays coupons within the holding period, there is a question how these should be incorporated into the holding period return calculation.

VII. Forward Contracts.

A. Definition.

1. Example: It is 2/15/95. You want to lock in an interest rate for the period 8/15/95 to 2/15/96. So agree today to invest an amount on 8/15/95 for 6 months at an interest rate (expressed as an APR with semi-annual compounding) of 6.38%. This rate is known as the forward rate available on 2/15/95 for the period 8/15/95 to 2/15/96.
   a. So if I agree on 2/15/95 to invest $1 on 8/15/95, I will receive $1+0.0638/2 = $1.0319 on 2/15/96.
   b. So I would have to agree on 2/15/95 to invest $(1/1.0319) = $0.9691 on 8/15/95 to get $1 on 2/15/96. Thus, $0.9691 is the discount factor associated with the forward contract.
2. Let $f_{t,t+\alpha}(0)$ be the α-period forward rate (expressed as an APR with semiannual compounding) available at time 0 for the period starting t periods after time 0.
   a. Example. You want to lock in an interest rate for the period 8/15/95 to 2/15/96. The relevant forward rate is $f_{½,1}$(Feb 95).
   b. If an investor knew at time 0 that she would have $1 M available to invest at time t and wanted a certain rate of return on a α period investment, then $f_{t,t+\alpha}(0)$ is the applicable yield expressed as an APR with semiannual compounding.
3. Let $d_{t,t+\alpha}(0)$ be the α-period forward contract discount factor available at time 0 for the period starting at time t.
   a. Example. You want to lock in an interest rate for the period 8/15/95 to 2/15/96. The relevant discount factor is $d_{½,1}$(Feb 95).
   b. So $d_{t,t+\alpha}(0)$ is the price to be paid at t for $1 delivered at (t+α) which can be locked in at time 0.
   c. By entering into a forward contract at time 0 to lock in a α-period rate at time t, the investor agrees to:
      (1) to invest \$ $d_{t,t+\alpha}(0)$ at time t; and,
      (2) receives \$ 1 for certain at time (t+α).

4. It is possible to go from $d_{t,t+\alpha}(0)$ to $f_{t,t+\alpha}(0)$ and vice versa using analogous formulas to those for discount bonds:

$$f_{t,t+\alpha}(0) = 2 \left\{ \left[\frac{1}{d_{t,t+\alpha}(0)} \right]^{1/(2\alpha)} - 1 \right\} \Leftrightarrow d_{t,t+\alpha}(0) = \frac{1}{\left[1 + \frac{f_{t,t+\alpha}(0)}{2}\right]^{2\alpha}} .$$

   a. Example (cont):
      (1) Can calculate the forward rate (expressed as an APR with semi-annual compounding) available on 2/15/95 for the period from 8/15/95 to 2/15/96 using the relevant discount factor for that forward rate $d_{½,1}(\text{Feb 95}) = 0.9691$:

$f_{½,1}(\text{Feb 95}) = 2 \times \{1/0.9691 - 1\} = 6.38\%$.

      (2) Can calculate the discount factor associated with the forward rate available on 2/15/95 for the period from 8/15/95 to 2/15/96 using the forward rate $f_{½,1}(\text{Feb 95}) = 6.38\%$:

$$d_{½,1}(Feb\ 95) = \frac{1}{\left[1 + \frac{0.0638}{2}\right]^{1}} = 0.9691.$$

5. Why is there a need for forward contracts.
   a. Because the yield curve shifts through time in a manner that is not totally predictable.
   b. Example.
      (1) Using a forward contract on 2/15/95, an investor can lock in an APR of 6.38% over the period 8/15/95 to 2/15/96.
      (2) If the investor waits until 8/15/95 to invest for the six month period, she may get a higher or lower rate.
      (3) Given the yield curve on 8/15/95 above, the investor would only have received an APR of 4.082% (YTM on the Feb 96 strip as of 8/15/95).
      (4) Investor may not want this uncertainty.

B. Relation between Forward Contract Discount Factors and Discount Bond Discount Factors.

1. Example (cont):

a. Can we create the forward rate contact in which we lock in a rate over the period *Aug 95* to *Feb 96* using discount bonds available on 2/15/95.

(1) The answer is yes.

(2) Use the *Aug 95* and the *Feb 96* strips.

b. Thus should be able to use the discount bond discount factors

$d_{½}$ (Feb 95) = 0.97; and
d_1 (Feb 95) = 0.94.

to get the forward contract discount factor $d_{½,1}$(Feb 95).

c. Two steps.

(1) First, buy a Feb 96 strip with a face value of \$1. This will cost d_1 (Feb 95) = 0.94 on 2/15/95.

(2) Second, sell d_1 (Feb 95) = 0.94 worth of the Aug 95 strip on 2/15/95; so total cash flow on the 2/15/95 is 0. The face value of the Aug 95 strips sold is

d_1 (Feb 95)/$d_{½}$ (Feb 95) = 0.94/0.97 = 0.9691.

(3) What are the cash flows.

2/15/95	8/15/95	2/15/96	
0	½	1	
-0.94		1	BUY \$1 face value Feb 96 strip
0.97 x 0.94/0.97	-0.94/0.97		SELL \$0.94/0.97 face value Aug 95 strip
0	-0.94/0.97	1	BUY synthetic forward contract

(4) So the law of one price implies that

$d_{½,1}$(Feb 95) = d_1 (Feb 95)/$d_{½}$ (Feb 95) = 0.94/0.97 = 0.9691.

2. In general,

$d_{t,t+\alpha}(0) = d_{t+\alpha}(0)/d_t(0)$.

3. Example. What is the discount factor associated with the forward contract that locks in a rate on 2/15/95 for the 1 year period 8/15/95 to 8/15/96? What is the forward rate expressed as an APR with semiannual compounding?

a. Need $d_{½,1½}$ (Feb 95). So using the formula,

$d_{½,1½}$ (Feb 95) = $d_{1½}$ (Feb 95) / $d_{½}$ (Feb 95) = 0.90/0.97 = 0.9278.

b. Get the forward rate $f_{½,1½}$ (Feb 95) as follows:

$f_{½,1½}$ (Feb 95) = 2 x $\{(1/0.9278)^{½} - 1\}$ = 7.63%.

C. Relation between Forward Rates and Discount Bond Yields.

1. The relation between forward contract discount factors and discount bond discount factors implies the following relation between forward rates and discount bond yields:

$$\left[1 + \frac{y_{t+\alpha}(0)}{2}\right]^{2(t+\alpha)} = \left[1 + \frac{y_t(0)}{2}\right]^{2t} \left[1 + \frac{f_{t,t+\alpha}(0)}{2}\right]^{2\alpha} .$$

2. Example. Bloomberg screen. 4/1/97. The yield on a 3-year discount bond $y_3(0)$ is 6.56%. The 1-year forward rate for the year starting in 3 years $f_{3,4}(0)$ is 6.90%. The yield on a 4-year discount bond $y_4(0)$ can be obtained as follows:

a. First, use above formula

$$\left[1 + \frac{y_4(0)}{2}\right]^{2\times 4} = \left[1 + \frac{y_3(0)}{2}\right]^{2\times 3} \left[1 + \frac{f_{3,4}(0)}{2}\right]^{2\times 1}$$

$$= \left[1 + \frac{0.0656}{2}\right]^{6} \left[1 + \frac{0.069}{2}\right]^{2} = 1.2988$$

b. Then obtain $y_4(0)$:

$$y_4(0) = 2\ \{1.2988^{1/8} - 1\} = 6.64\%$$

which agrees with the Bloomberg screen.

VIII. Theories of the Yield Curve.

A. Forward Rates determine the Yield Curve.

1. The relation between forward rates and the yield on discount bonds can be specialized to the following:

$$d_{t+½}(0) = d_t(0)\, d_{t,t+½}(0) \Leftrightarrow \left[1 + \frac{y_{t+½}(0)}{2}\right]^{2(t+½)} = \left[1 + \frac{y_t(0)}{2}\right]^{2t} \left[1 + \frac{f_{t,t+½}(0)}{2}\right]^{1} .$$

2. It follows that the shape of the yield curve at any point in time depends on the spot rate $y_{½}(0)$ together with the sequence of ½ year forward rates available at that time.
   a. If $f_{t,t+½}(0) > y_t(0)$ then $y_{t+½}(0) > y_t(0)$.
   b. If $f_{t,t+½}(0) < y_t(0)$ then $y_{t+½}(0) < y_t(0)$.
3. Note:
   a. an upward sloping yield curve implies that the sequence of ½ year forward rates are higher than the current spot rate $y_{½}(0)$.
   b. a downward sloping yield curve implies that the sequence of ½ year forward rates are all lower than the current spot rate $y_{½}(0)$.

B. How are forward rates determined?

1. Expectations Theory.

a. According to this theory, today's forward rate for any future ½ year period is equal to the expected interest rate over that ½ year period.

b. So

$f_{t,t+½}(0) = E_{\text{at time } 0}[y_{½}(t)]$.

c. Idea is that if an investor does not care about uncertainty then the expected payoff at time (t+½) from the following two strategies should be the same:

(1) Buy \$1 of t-year discount bonds and commit to invest the proceeds for ½ a year at $f_{t,t+½}(0)$: the certain payoff is

$$[1 + \frac{y_t(0)}{2}]^{2t} [1 + \frac{f_{t,t+½}(0)}{2}]^{1} .$$

(2) Buy \$1 of t-year discount bonds and invest the proceeds for a ½ year at the prevailing spot rate at time t which will be $y_{½}(t)$: expected payoff is

$$[1 + \frac{y_t(0)}{2}]^{2t} [1 + \frac{E_{at\ time0}[y_{½}(t)]}{2}]^{1} .$$

d. This argument is not an arbitrage argument; it relies on individuals not caring about the uncertainty associated with the second strategy.

e. If the expectations hypothesis holds, then:

(1) an upward sloping yield curve implies that the market is expecting higher spot rates in the future.

(2) a downward sloping yield curve implies that the market is expecting lower spot rates in the future.

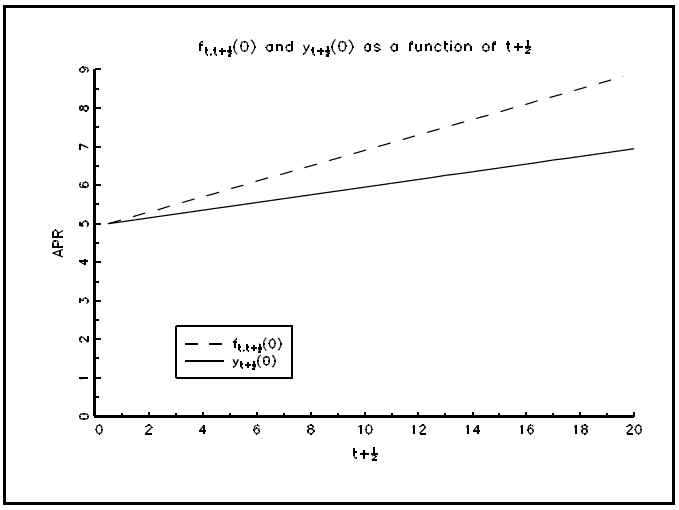
2. Liquidity Preference Theory.
   a. This theory says that forward rates embody a liquidity premia in addition to the expected future spot rate.
   b. So

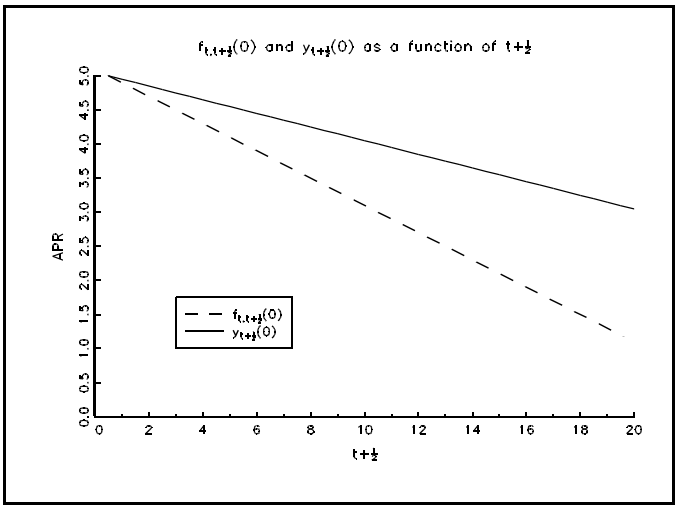
$$f_{t,t+½}(0) = E_{\text{at time } 0}[y_{½}(t)] + p_{t+½}(0)$$

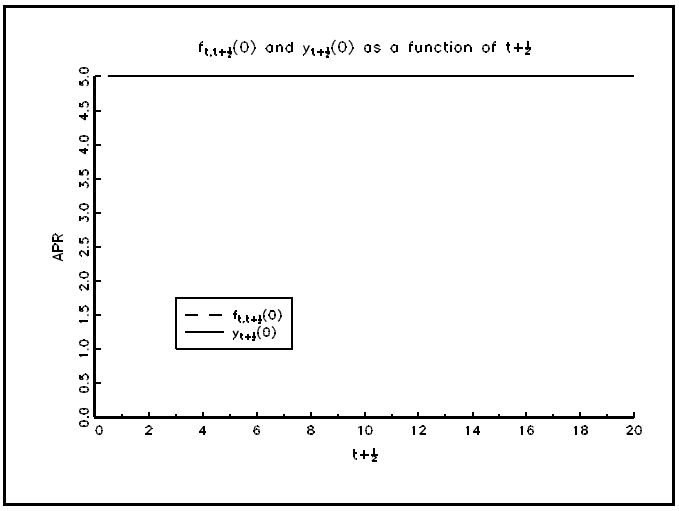
where $p_{t+½}(0)$ is the liquidity premium.

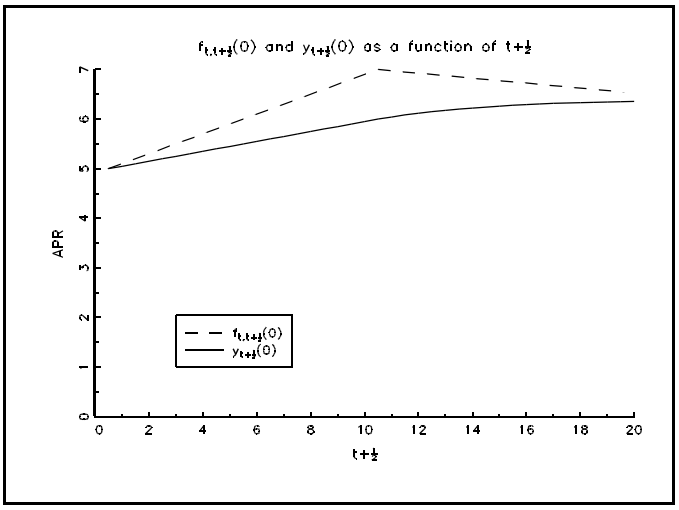
   c. Theory says that:
      (1) $p_{t+½}(0) > 0$ for all $t>0$.
   d. The idea is that individuals require a higher return to induce them to invest in less liquid, longer maturity bonds.
   e. If the liquidity preference hypothesis holds, then:
      (1) an upward sloping yield curve is consistent with the market expecting higher or lower spot rates in the future.
      (2) a downward sloping yield curve implies that the market is expecting lower spot rates in the future.

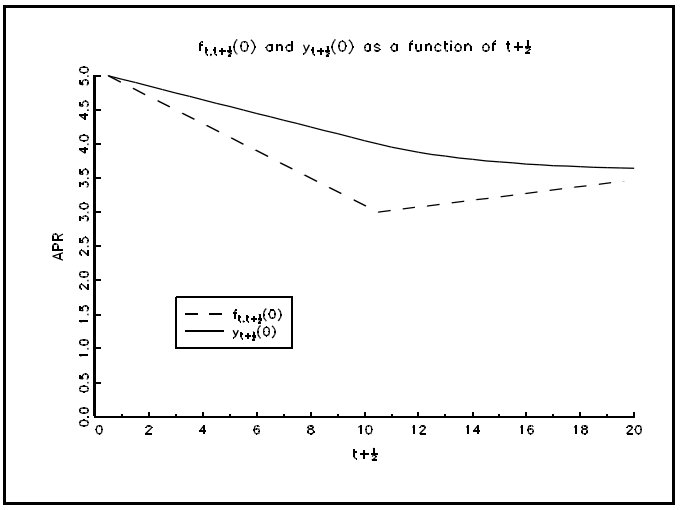
3. Preferred Habitat Theory.
   a. This theory says that yields on different securities are determined by the supply and demand for that security and that securities with similar maturities are not close substitutes.
   b. Theory says that:
      (1) $p_{t+½}(0)$ can be positive or negative for a given t depending on supply and demand for the (t+½)-period discount bond.
   c. If the preferred habitat hypothesis holds, then the slope of the yield curve tell you little about whether the market is expecting higher or lower spot rates in the future.

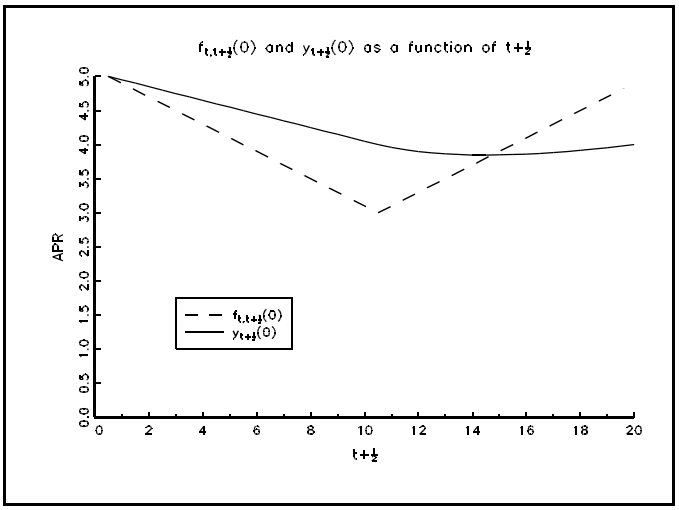

$f_{t,t+\frac{1}{2}}(0)$ and $y_{t+\frac{1}{2}}(0)$ as a function of $t+\frac{1}{2}$
APR
$t+\frac{1}{2}$
$f_{t,t+\frac{1}{2}}(0)$
$y_{t+\frac{1}{2}}(0)$

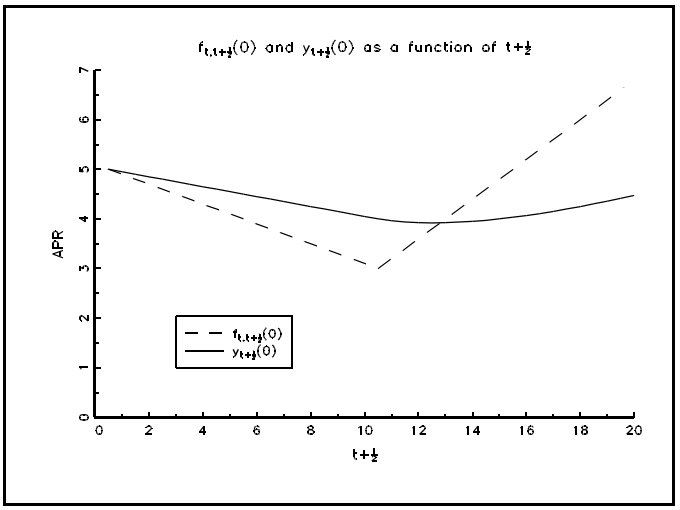

$f_{t,t+\frac{1}{2}}(0)$ and $y_{t+\frac{1}{2}}(0)$ as a function of $t+\frac{1}{2}$
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$t+\frac{1}{2}$
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$t+\frac{1}{2}$
$f_{t,t+\frac{1}{2}}(0)$
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$f_{t,t+\frac{1}{2}}(0)$ and $y_{t+\frac{1}{2}}(0)$ as a function of $t+\frac{1}{2}$
APR
$t+\frac{1}{2}$
$f_{t,t+\frac{1}{2}}(0)$
$y_{t+\frac{1}{2}}(0)$

$$d_{t+½}(0) = d_t(0)\, d_{t,t+½}(0) \Leftrightarrow \left[1 + \frac{y_{t+½}(0)}{2}\right]^{2(t+½)} = \left[1 + \frac{y_t(0)}{2}\right]^{2t} \left[1 + \frac{f_{t,t+½}(0)}{2}\right]^{1} .$$

$$\left[1 + \frac{y_{½}(0)}{2}\right] .$$

$$\left[1 + \frac{y_1(0)}{2}\right]^2 = \left[1 + \frac{y_{½}(0)}{2}\right] \left[1 + \frac{f_{½,1}(0)}{2}\right] .$$

$$\left[1 + \frac{y_{1½}(0)}{2}\right]^3 = \left[1 + \frac{y_1(0)}{2}\right]^2 \left[1 + \frac{f_{1,1½}(0)}{2}\right] .$$

$$\left[1 + \frac{y_2(0)}{2}\right]^4 = \left[1 + \frac{y_{1½}(0)}{2}\right]^3 \left[1 + \frac{f_{1½,2}(0)}{2}\right] .$$

$$\left[1 + \frac{y_{2½}(0)}{2}\right]^5 = \left[1 + \frac{y_2(0)}{2}\right]^4 \left[1 + \frac{f_{2,2½}(0)}{2}\right] .$$

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