Lecture 10-11: CAPM.

I. Reading
   A. BKM, Chapter 9, Section 9.1.
   B. BKM, Chapter 10, Section 10.1 and 10.2.

II. Market Portfolio.
   A. Definition: The market portfolio M is the portfolio of all risky assets in the economy each asset weighted by its value relative to the total value of all assets.
   B. Economy: N risky assets and J individuals.
   C. Weight of asset i in the market portfolio ($\omega_{i,M}$) is given by:

$$\omega_{i,M} = \frac{V_i}{V_M}$$

where

V_i is the market value of the ith risky asset;
$V_M = V_1 + ... + V_N$ is the total value of all risky assets in the economy.

   D. One Formula for the Return on the Market Portfolio:

$$R_M = \omega_{1,M} R_1 + ... + \omega_{N,M} R_N$$

where

R_M is the return on the value weighted market portfolio;
R_i is the return on the ith risky asset, i=1,2,...,N;

E. Example: Suppose there are only 2 individuals and 3 risky assets in the economy.

1. Individual 1 invests \$80000 in risky assets of which \$40000 is in asset 1, \$30000 in asset 2 and \$10000 in asset 3. Individual 2 invests \$20000 in risky assets of which \$6000 is in asset 1, \$12000 is in asset 2 and \$2000 is in asset 3.
2. Return on asset 1 is 10%. Return on asset 2 is 20%. Return on asset 3 is -10%.

	Individual 1		Individual 2		Market	
Asset i	$V_{i,p1}$	$\omega_{i,p1}$	$V_{i,p2}$	$\omega_{i,p2}$	V_i	$\omega_{i,M}$
1	40000	0.500	6000	0.3	46000	0.46
2	30000	0.375	12000	0.6	42000	0.42
3	10000	0.125	2000	0.1	12000	0.12
Total	80000	1.000	20000	1.000	100000	1.000

3. What is the market value of asset 1? $V_1 = 40000+6000 = 46000$.

4. What is the weight of asset 1 in the market portfolio?

$\omega_{i,M} = 46000/100000 = 0.46$.

5. What is the return on the market portfolio?

$$R_M = \omega_{1,M} R_1 + \omega_{2,M} R_2 + \omega_{3,M} R_3 = 0.46 \text{ x } 10\% + 0.42 \text{ x } 20\% + 0.12 \text{ x } -10\% = 11.8\%$$

6. What is the return on each individual's portfolio (p1 and p2)?

1: $R_{p1} = \omega_{1,p1} R_1 + \omega_{2,p1} R_2 + \omega_{3,p1} R_3 = 0.5 \text{ x } 10\% + 0.375 \text{ x } 20\% + 0.125 \text{ x } -10\% = 11.25\%$

2: $R_{p2} = \omega_{1,p2} R_1 + \omega_{2,p2} R_2 + \omega_{3,p2} R_3 = 0.3 \text{ x } 10\% + 0.6 \text{ x } 20\% + 0.1 \text{ x } -10\% = 14\%$

7. But can see that the market portfolio can be formed by adding together the portfolios of the two individuals. Can think of the market portfolio as a portfolio with 80% (80000/100000) invested in individual 1's portfolio and 20% in individual 2's portfolio. Thus, can calculate the market portfolio's return:

$$R_M = 0.8\ R_{p1} + 0.2\ R_{p2} = 0.8 \text{ x } 11.25\% + 0.2 \text{ x } 14\% = 11.8\%$$

F. Another Formula for Market Return: The market portfolio can also be thought of

as a portfolio of individuals' risky asset portfolios where the weights are the value of each individual's portfolio relative to the total value of all assets.

$$R_M = \frac{W_1}{V_M} R_{p1} + \ldots + \frac{W_J}{V_M} R_{pJ}$$

where

R_{pj} is the return on the jth individual's risky portfolio, j=1,2,...,J;
W_{pj} is the market value of the jth individual's risky asset portfolio;
$V_M = W_1 + ... + W_J$.

G. How to calculate the market value of a firm's equity:

1. Formula:

$$V_i = n_i p_i$$

where:

n_i is the number of shares of equity i outstanding;
p_i is the price of a share of i.

2. Example: IBM has 517.546M shares outstanding at a price of $143.875 at close Monday 2/24/97. So

V_{IBM} = 517.546M x $143.875 = $74461.93M.

III. CAPM World: Assumptions.

A. All individuals care only about expected return and standard deviation of return.

B. Individuals agree on the opportunity set of assets available.

C. Individuals can borrow and lend at the one riskfree rate.

D. Individuals can trade costlessly, can sell short any asset, face zero taxes, can hold any fraction of an asset and are price takers. This assumption is known as the perfect capital markets assumption.

IV. Portfolio Choice in a CAPM World.

A. All individuals want to hold a combination of the riskless asset and the tangency portfolio.

B. Example (cont): Suppose a CAPM wold exists in our 2 individual, 3 asset economy. The tangency portfolio invests 30% in asset 1, 50% in asset 2 and 20% in asset 3. Individual 1 invests \$80000 in the tangency portfolio and individual 2 invests \$20000 in the tangency portfolio.

	Individual 1		Individual 2		Market	
Asset i	$V_{i,p1}$	$\omega_{i,p1}$	$V_{i,p2}$	$\omega_{i,p2}$	V_i	$\omega_{i,m}$
1	24000	0.3	6000	0.3	30000	0.3
2	40000	0.5	10000	0.5	50000	0.5
3	16000	0.2	4000	0.2	20000	0.2
Total	80000	1.0	20000	1.0	100000	1.0

Since both investors hold the tangency portfolio as their risky asset portfolio, can see that the market portfolio of risky assets must be the tangency portfolio.

C. Since everyone holds the same risky portfolio and the market portfolio is a weighted average of individuals' portfolios, all individuals must be holding the market as their risky portfolio; the market portfolio is the tangency portfolio.

D. So everyone holds some combination of the value weighted market portfolio M and the riskless asset.

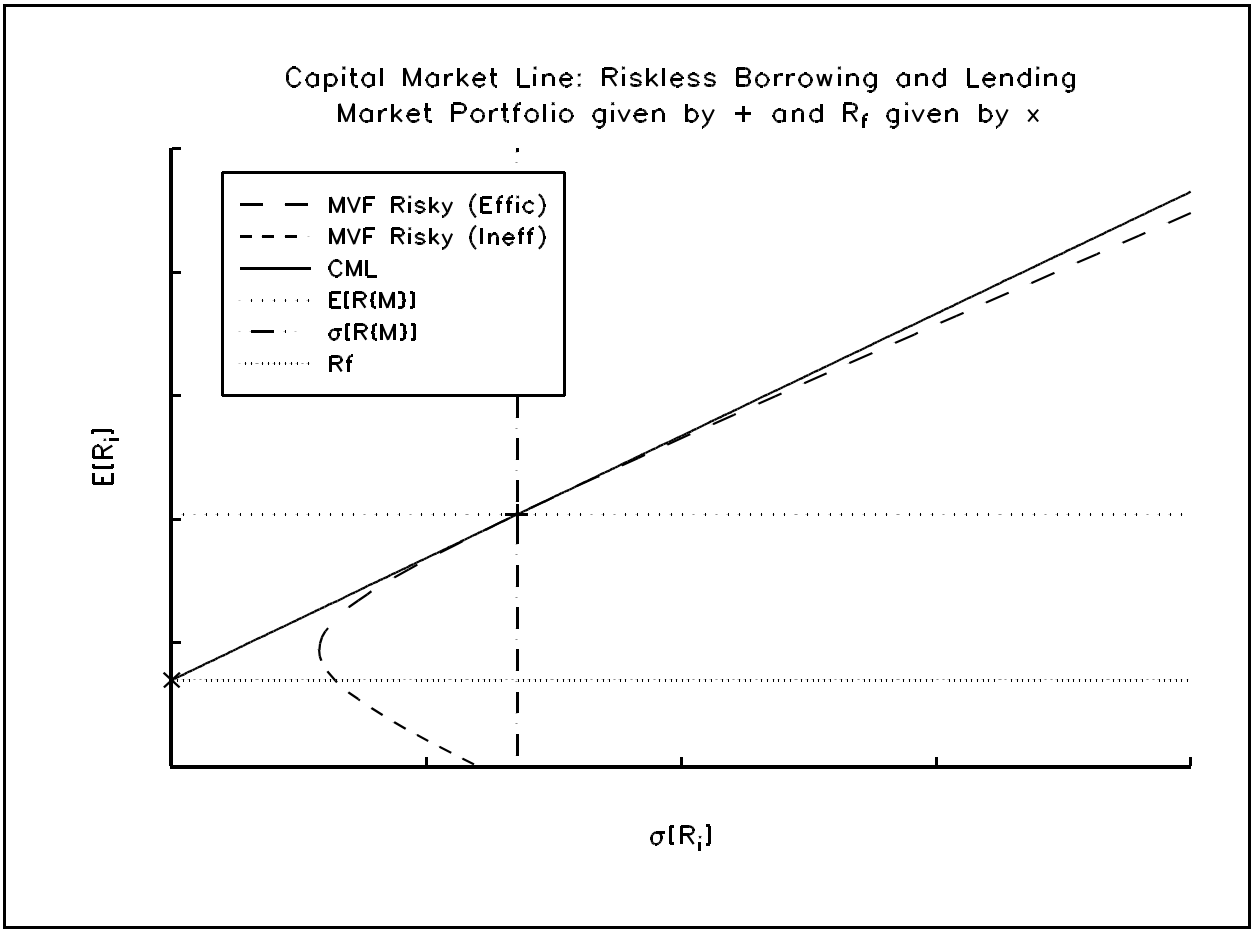
E. Capital Market Line (CML).

1. The CAL which is obtained by combining the market portfolio and the riskless asset is known as the Capital Market Line (CML) and has the following formula:

$$CML: \quad E[R_{ef}] = R_f + \frac{E[R_M] - R_f}{\sigma[R_M]} \sigma[R_{ef}]$$

where *ef* is a portfolio that is a combination of the riskless asset and the market portfolio.

2. Portfolios that lie on the CML are known as efficient portfolios and have the following properties:
   a. Only assets which are a combination of the riskless asset and the market portfolio lie on the CML.
   b. For any individual, the portfolio she holds lies on the CML.
   c. Any portfolio on the CML has correlation of 1 with the market portfolio since it is a combination of the riskless asset and the market.



V. Minimum Variance Mathematics.

A. The following results can be shown to hold *mathematically* and contain *no* economics.

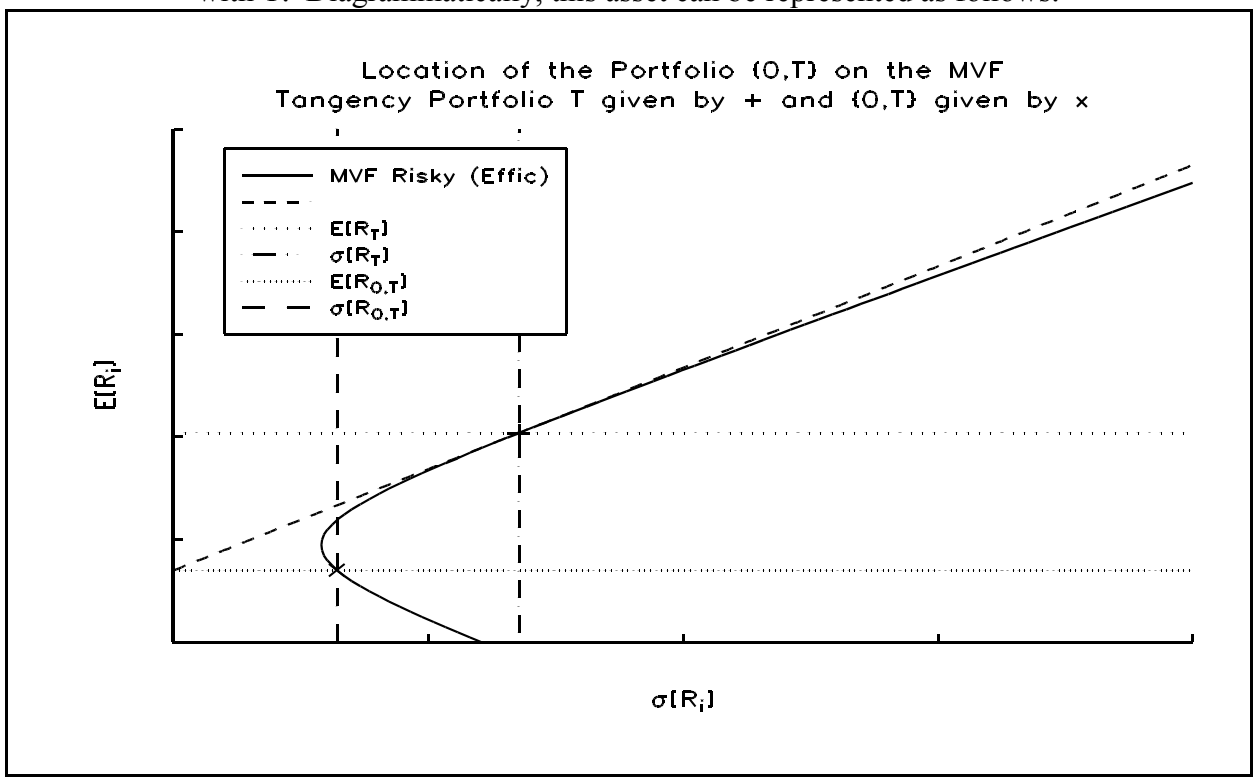
B. Suppose unlimited short selling is allowed. Saying T lies on the minimum variance frontier for N risky assets i =1, 2, ..., N is equivalent to saying that the following holds for any portfolio p of the N risky asset returns:

$$E[R_p] = E[R_{0,T}] + \{E[R_T] - E[R_{0,T}]\}\ \beta_{p,T}$$

where

$$\beta_{p,T} = \frac{\sigma[R_p,\ R_T]}{\sigma[R_T]^2}$$

and asset {0,T} is the asset on the minimum variance frontier that is uncorrelated with T. Diagrammatically, this asset can be represented as follows:



C. Suppose unlimited short selling is allowed. If $T_1, T_2,...,T_K$ lie on the MVF for the N risky assets then any portfolio formed from these K portfolios also lies on the minimum variance frontier.

VI. Individual Assets in a CAPM World.

A. Importance: Why care about the expected return for an individual asset?

1. Stock Valuation: What discount rate do we use to discount the expected cash flows from the stock?
2. Capital Budgeting: What rate do we use as the cost of equity capital?

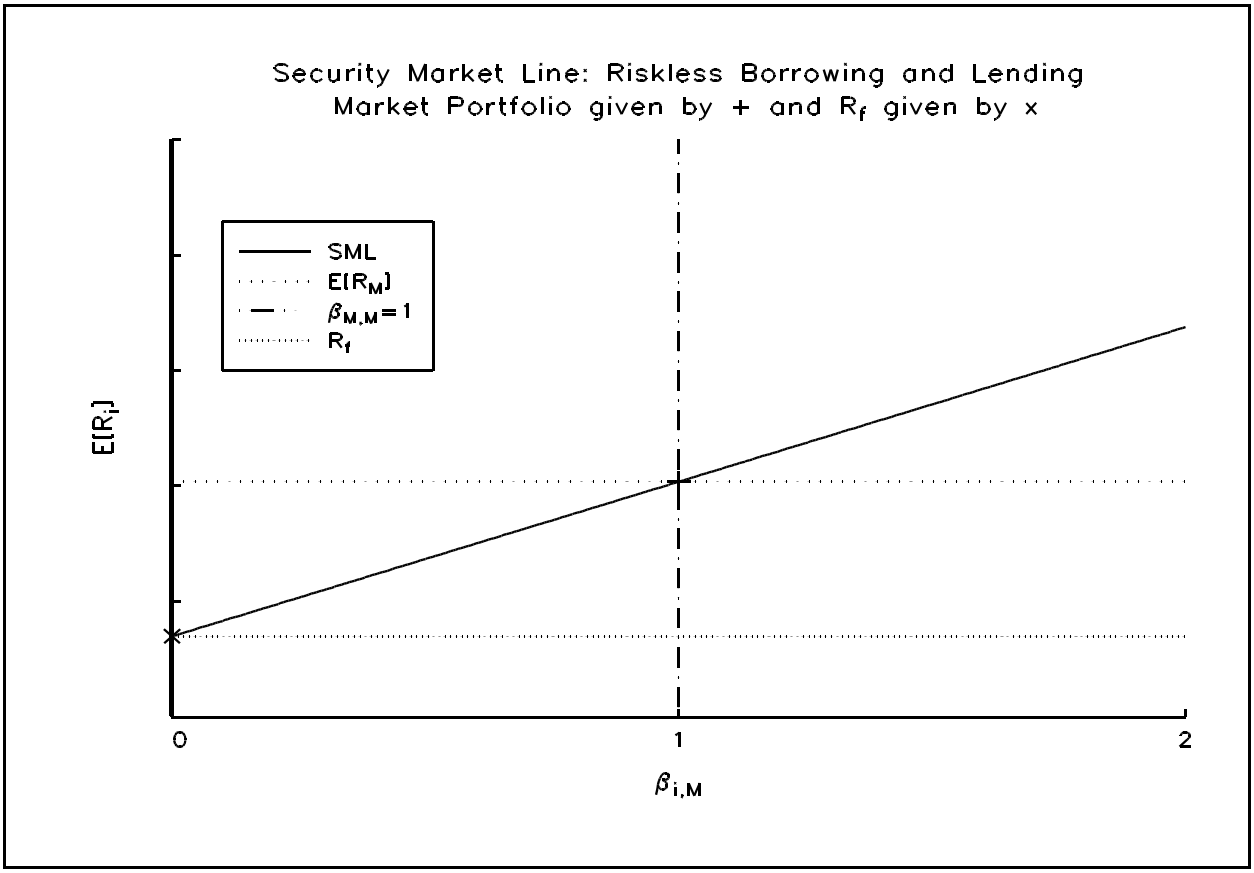
B. Main Result.

1. Since the market portfolio lies on the MVF for the N risky assets, the mathematical result described above implies that the following relation ship holds for *any* portfolio p formed from the N risky assets:

$$E[R_p] = E[R_{0,M}] + \{E[R_M] - E[R_{0,M}]\}\ \beta_{p,M} \ .$$

2. Can see geometrically that $E[R_{0,M}] = R_f$. So all assets lie on the following line called the Security Market Line:

$$SML: \quad E[R_p] = R_f + \{E[R_M] - R_f\}\ \beta_{p,M} \ .$$



C. Properties of Beta:

1. The Beta of the riskless asset is 0: $\beta_{f,M} = \sigma[R_f, R_M] / \sigma[R_M]^2 = 0$.
2. The Beta of the minimum variance portfolio uncorrelated with the market is 0: $\beta_{\{0,M\},M} = \sigma[R_{0,M}, R_M] / \sigma[R_M]^2 = 0$.
3. The Beta of the market is 1: $\beta_{M,M} = \sigma[R_M, R_M] / \sigma[R_M]^2 = 1$.
4. The Beta of a portfolio is a weighted average of the Betas of the assets that comprise the portfolio where the weights are those of the assets in the portfolio. So if the portfolio return is given by:

$$R_p = \omega_{f,p} R_f + \omega_{1,p} R_1 + \omega_{2,p} R_2 + ...+ \omega_{K,p} R_K$$

then the portfolio's Beta is given by

$$\beta_{p,M} = \omega_{f,p} \beta_{f,M} + \omega_{1,p} \beta_{1,M} + \omega_{2,p} \beta_{2,M} + ...+ \omega_{K,p} \beta_{K,M} = \omega_{1,p} \beta_{1,M} + \omega_{2,p} \beta_{2,M} + ...+ \omega_{K,p} \beta_{K,M} .$$

D. SML holds for Portfolios of Risky Assets and the Riskless Asset.

1. Since the SML relation holds for risky asset portfolios and for the riskless asset ($\beta_{f,M} = 0 \Rightarrow E[Rf] = Rf + 0$ using SML), it also holds for portfolios that contain the riskless asset as well as risky assets:

$$SML: \quad E[R_p] = R_f + \{E[R_M] - R_f\} \beta_{p,M} .$$

VII. Intuition for the SML ($E[R_p]$ depending on $\beta_{p,M}$).

A. Slope of the CML.

1. Everybody holds portfolios which lie on the CML:

$$CML: \quad E[R_{ef}] = R_f + \frac{E[R_M] - R_f}{\sigma[R_M]} \sigma[R_{ef}]$$

2. The slope of the CML depends on $\{E[R_M]-R_f\}$ relative to $\sigma[R_M]$.

B. Decomposing the Variance of the Market Portfolio.

1. It can be shown that $\sigma[R_M]^2$ can be written as a weighted average of the covariance of the individual assets with the market portfolio:

$$\sigma^2[R_M] = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_{i,M} \, \omega_{j,M} \, \sigma[R_i, R_j]$$

$$= \sum_{i=1}^{N} \omega_{i,M} \, \sigma[R_i, R_M]$$

2. So $\sigma[R_i,R_M]$ measures the contribution of asset i to $\sigma[R_M]^2$.

3. Since

$$\beta_{i,M} = \frac{\sigma[R_i, R_M]}{\sigma[R_M]^2}$$

it follows that $\beta_{i,M}$ measures the contribution of asset i to $\sigma[R_M]^2$ as a fraction of the market portfolio's variance.

4. So it makes sense that $E[R_i]$ depends on $\beta_{i,M}$: why the relation is linear is less clear and depends in part on the mathematical results stated earlier. BKM also contains a discussion about why the relation is linear on pgs 255-259.

VIII. CML vs SML.

A. All assets lie on the SML yet only efficient portfolios which are combinations of the market portfolio and the riskless asset lie on the CML

B. How can this be?

1. First note that since by definition

$\sigma[R_p,R_M] = \rho[R_p,R_M]\ \sigma[R_p]\ \sigma[R_M]$

it follows that

$$\beta_{p,M} = \frac{\sigma[R_p,\ R_M]}{\sigma[R_M]^2} = \frac{\rho[R_p,\ R_M]\ \sigma[R_p]\ \sigma[R_M]}{\sigma[R_M]^2} = \frac{\rho[R_p,\ R_M]\ \sigma[R_p]}{\sigma[R_M]} \ .$$

2. Thus, the SML can be written

$$SML: \quad E[R_p] = R_f + \frac{E[R_M] - R_f}{\sigma[R_M]}\ \{\rho[R_p,R_M]\ \sigma[R_p]\} \ .$$

3. Comparing this equation to the CML

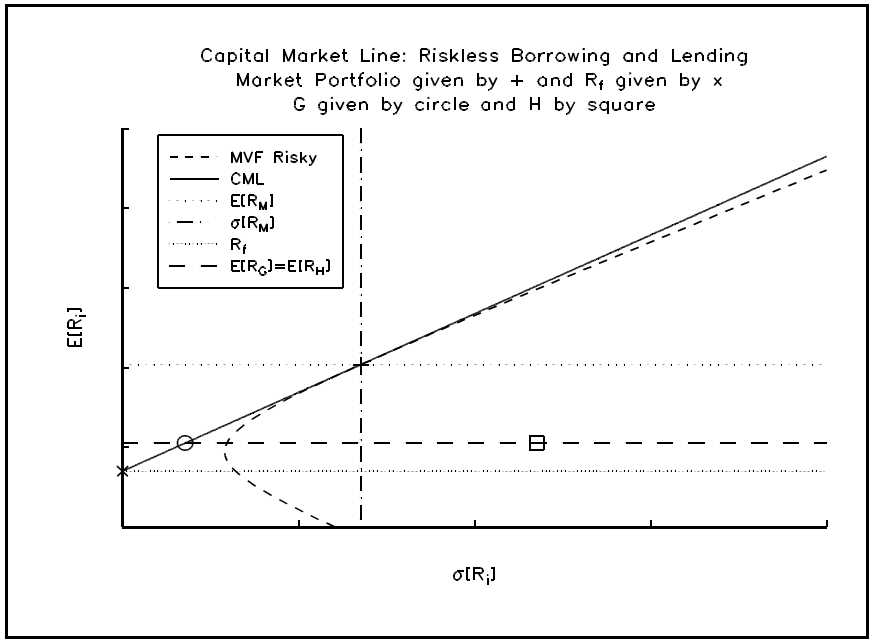
$$CML: \quad E[R_{ef}] = R_f + \frac{E[R_M] - R_f}{\sigma[R_M]}\ \sigma[R_{ef}]$$

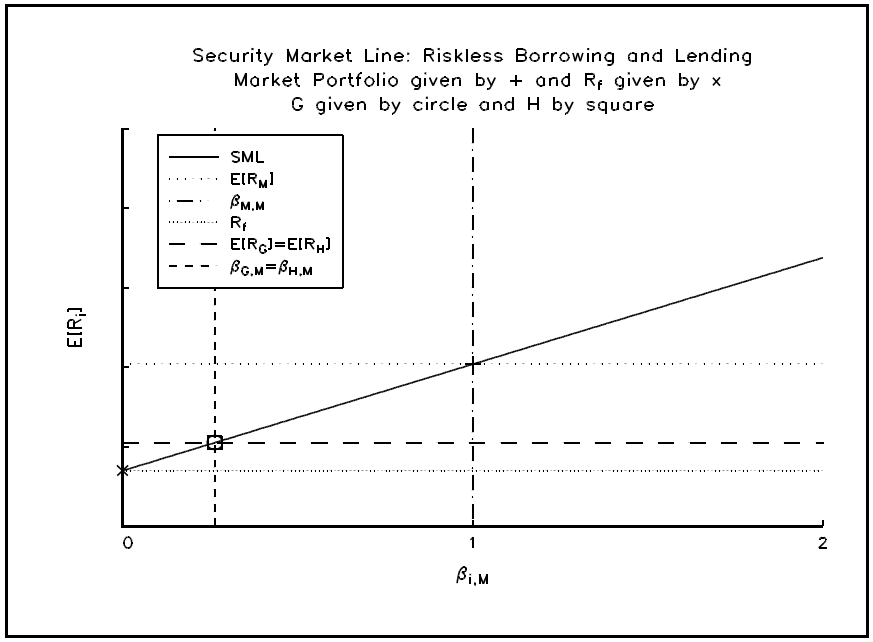
it can be seen that:

a. an asset p lies on the SML and the CML if $\rho[R_p,R_M]=1$.

b. an asset p only lies on the SML and is not a combination of the riskless asset and the market portfolio if $\rho[R_p,R_M]<1$.

C. Example: Suppose the CAPM holds. Two assets G and H have the same Beta with respect to the market: $\beta_{G,M} = \beta_{H,M}$. Since all assets including G and H lie on the SML, both have the same expected return: $E[R_G] = E[R_H]$. But G is a combination of the market portfolio and the riskless asset and so lies on the CML while H lies to the right of the CML having a higher standard deviation than G: $\sigma[R_G] < \sigma[R_H]$. Further $\rho[R_G, R_M] = 1$ while $\rho[R_H, R_M] < 1$.





IX. Example Problem .Assume that the CAPM holds in the economy. The following data is available about the market portfolio, the riskless rate and two assets, G and H. Remember $\beta_{p,M} = \sigma[R_p , R_M]/(\sigma[R_M]^2)$.

Asset i	$E[R_i]$	$\sigma[R_i]$	$\beta_{i,M}$
M (market)	0.13	0.10	
G		0.05	0.5
H		0.08	0.5

$R_f = 0.05$.

A. What is the expected return on asset G (i.e., $E[R_G]$)?

All assets plot on the SML:

$E[R_p] = R_f + \beta_{p,M} \{E[R_M] - R_f\}$

So

$E[R_G] = R_f + \beta_{G,M} \{E[R_M] - R_f\} = 0.05 + 0.5\{0.13-0.05\} = 0.09.$

B. What is the expected return on asset H (i.e., $E[R_H]$)?

Similarly,

$E[R_H] = R_f + \beta_{H,M} \{E[R_M] - R_f\} = 0.05 + 0.5\{0.13-0.05\} = 0.09.$

C. Does asset G plot:

1. on the SML (security market line)?

Yes.

2. on the CML (capital market line)?

Formula for the CML:

$E[R_{ef}] = R_f + \sigma[R_{ef}] \{E[R_M] - R_f\}/\sigma[R_M]$.

For G,

$R_f + \sigma[R_G] \{E[R_M] - R_f\}/\sigma[R_M] = 0.05 + 0.05\{0.13-0.05\}/0.10 = 0.09 = E[R_G]$

as required for G to lie on the CML.

D. Does asset H plot:

1. on the SML?

Yes.

2. on the CML?

For H,

$R_f + \sigma[R_H] \{E[R_M] - R_f\}/\sigma[R_M] = 0.05 + 0.08\{0.13-0.05\}/0.10 = 0.114 > E[R_H]$

and so H does not lie on CML.

E. Could any investor be holding asset G as her entire portfolio?

Yes since it lies on the CML.

F. Could any investor be holding asset H as her entire portfolio?

No since it does not lie on the CML.

G. What is the correlation of asset G with the market portfolio?

Recall

$\beta_{p,M} = \rho[R_p, R_M] \sigma[R_p] / \sigma[R_M]$

which implies

$\rho[R_p, R_M] = \beta_{p,M} \sigma[R_M] / \sigma[R_i]$.

So, for G,

$\rho[R_G, R_M] = \beta_{G,M} \sigma[R_M] / \sigma[R_G] = (0.5x0.10)/0.05 = 1$.

H. What is the correlation of asset H with the market portfolio?

Similarly, for H,

$\rho[R_H, R_M] = \beta_{H,M} \sigma[R_M] / \sigma[R_H] = (0.5x0.10)/0.08 = 0.625$.

I. Can anything be said about the composition of asset G (i.e., what assets make up asset G)?

Since G lies on the CML, it must be some combination of the market portfolio and the riskless asset.

J. Can anything be said about the composition of asset H?

No.

X. More Intuition for the SML ($E[R_p]$ depending on $\beta_{p,M}$).

A. Think of running a regression of R_p on R_M.

$R_p = \mu_{p,M} + \beta_{p,M} R_M + e_{p,M}$

1. The $\mu_{p,M}$ and $\beta_{p,M}$ which minimize $E[e_{p,M}^2]$ are known as regression coefficients and are given by:

$$\beta_{p,M} = \frac{\sigma[R_p, R_M]}{\sigma[R_M]^2}; \quad and, \quad \mu_{p,M} = E[R_p] - \beta_{p,M} E[R_M] .$$

2. So the slope coefficient from a regression of R_p on R_M is the Beta of asset i with respect to the market portfolio.
3. Further, it can be shown that $\sigma[R_M, e_{p,M}] = 0$.

B. Decomposing the Variance of asset p:

$$\begin{aligned}\sigma[R_p]^2 &= \sigma[\mu_{p,M} + \beta_{p,M} R_M + e_{p,M}]^2 \\ &= \beta_{p,M}^2 \sigma[R_M]^2 + \sigma[e_{p,M}]^2 + 2\beta_{p.M} \sigma[R_M, e_{p,M}] \\ &= \beta_{p,M}^2 \sigma[R_M]^2 + \sigma[e_{p,M}]^2\end{aligned}$$

since $\sigma[R_M, e_{p,M}] = 0$.

C. In the context of holding the market portfolio as your risky portfolio, the first term represents the undiversifiable risk of asset p while the second term represents the risk which is diversified away when asset p is held in the market portfolio.

D. It can be seen that portfolio p's undiversifiable risk depends on $\beta_{p,M}$.

E. Hence it makes sense that in a CAPM setting $E[R_p]$ depends on $\beta_{p,M}$ since every individual holds some combination of the market portfolio and the riskless asset.

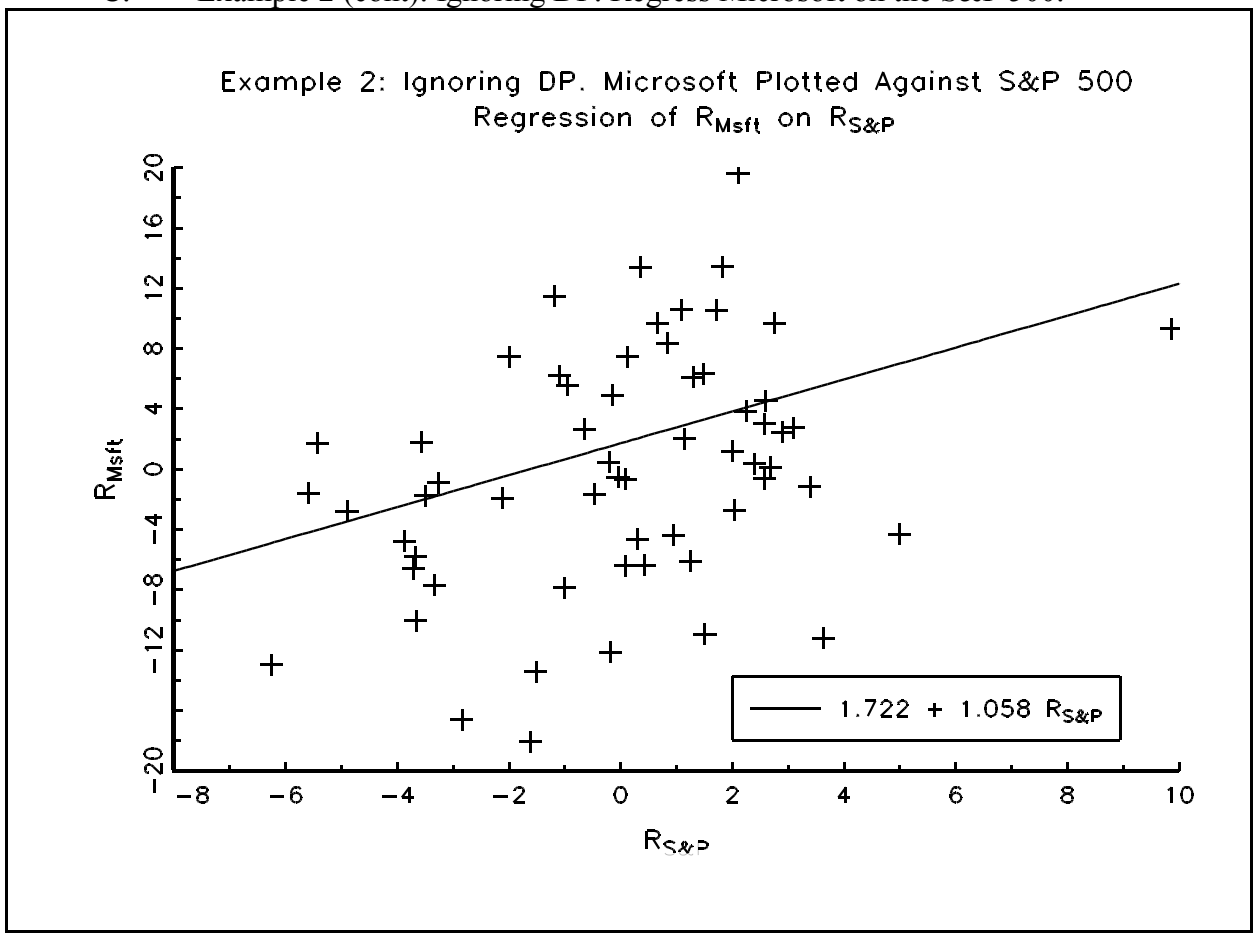
XI. Beta Estimation.

A. If return distributions are the same every period, then can use a past series of returns to run regressions of R_p on R_M to obtain an estimate of $\beta_{p,M}$.

B. Market Portfolio Proxy.

1. Can not observe the return on the market portfolio.
2. Use the S&P 500 index as a proxy.
3. Why?
   a. S&P 500 contains 500 stocks chosen for "representativeness".
   b. S&P 500 is value-weighted.

C. Example 2 (cont): Ignoring DP. Regress Microsoft on the S&P 500.



D. Empirical evidence suggests that over time the Betas of stock move toward the average Beta of 1. For this reason, a raw estimate of Beta is often adjusted using the following formula: $\beta_{adj} = w\,\beta_{est} + (1\text{-}w)\,1$.

XII. Application: Can the CAPM Explain the Small Firm Effect?

A. Using Historical Data 1/27-12/95.

1. Use S&P to proxy for the market portfolio.
2. $E[R_f]$ =0.30% per month.

B. Small Firm Effect restated.

1. $E[R_{Small}]$ = 1.37% per month.
2. $E[R_{S\&P}]$ = 1.00% per month.
3. $E[R_{Small}] > E[R_{S\&P}]$.

C. Is $E[R_{Small}] > E[R_{S\&P}]$ due to $\beta_{Small,S\&P} > 1$ and Small lying on SML?

1. $\beta_{Small,S\&P} = 1.297$.
2. SML says:

$$E[R_{Small}] - R_f = \beta_{Small,S\&P} \{E[R_{S\&P}] - R_f\}.$$

3. Yet:

$$E[R_{Small} - R_f] = 1.37 - 0.30 = 1.07 > \beta_{Small,S\&P} \{E[R_{S\&P} - R_f]\} = 1.297 \{1.00 - 0.30\} = 0.91$$

4. So the small firm portfolio lies above the SML.

D. The small firm effect is an example of the poor performance of the CAPM empirically.