Lecture 9: Portfolio Management - N Risky Assets and a Riskless Asset

I. Reading.
A. BKM, Chapter 8, Sections 8.4 and 8.5 and Appendix 8.A.

II. Standard Deviation of Portfolio Return: N Risky Assets.
1. Formula.

$$\sigma^2[R_p(t)] = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_{i,p}\ \omega_{j,p}\ \sigma[R_i(t),\ R_j(t)]$$

where

$\sigma[R_i(t), R_j(t)]$ is the covariance of asset i's return and asset j's return in period t;
$\omega_{i,p}$ is the weight of asset i in the portfolio p;
$\sigma^2[R_p(t)]$ is the variance of return on portfolio p in period t.

2. The formula says that $\sigma^2[R_p(t)]$ is equal to the sum of the elements in the following N x N matrix.

	1	2	... j ...	N-1	N
1	$\omega_{1,p}\ \omega_{1,p}$ $\sigma[R_1, R_1]$	$\omega_{1,p}\ \omega_{2,p}$ $\sigma[R_1, R_2]$	$\omega_{1,p}\ \omega_{j,p}$ $\sigma[R_1, R_j]$	$\omega_{1,p}\ \omega_{N-1,p}$ $\sigma[R_1, R_{N-1}]$	$\omega_{1,p}\ \omega_{N,p}$ $\sigma[R_1, R_N]$
2	$\omega_{2,p}\ \omega_{1,p}$ $\sigma[R_2, R_1]$	$\omega_{2,p}\ \omega_{2,p}$ $\sigma[R_2, R_2]$	$\omega_{2,p}\ \omega_{j,p}$ $\sigma[R_2, R_j]$	$\omega_{2,p}\ \omega_{N-1,p}$ $\sigma[R_2, R_{N-1}]$	$\omega_{2,p}\ \omega_{N,p}$ $\sigma[R_2, R_N]$
...i...	$\omega_{i,p}\ \omega_{1,p}$ $\sigma[R_i, R_1]$	$\omega_{i,p}\ \omega_{2,p}$ $\sigma[R_i, R_2]$	$\omega_{i,p}\ \omega_{j,p}$ $\sigma[R_i, R_j]$	$\omega_{i,p}\ \omega_{N-1,p}$ $\sigma[R_i, R_{N-1}]$	$\omega_{i,p}\ \omega_{N,p}$ $\sigma[R_i, R_N]$
N-1	$\omega_{N-1,p}\ \omega_{1,p}$ $\sigma[R_{N-1}, R_1]$	$\omega_{N-1,p}\ \omega_{2,p}$ $\sigma[R_{N-1}, R_2]$	$\omega_{N-1,p}\ \omega_{j,p}$ $\sigma[R_{N-1}, R_j]$	$\omega_{N-1,p}\ \omega_{N-1,p}$ $\sigma[R_{N-1},R_{N-1}]$	$\omega_{N-1,p}\ \omega_{N,p}$ $\sigma[R_{N-1}, R_N]$
N	$\omega_{N,p}\ \omega_{1,p}$ $\sigma[R_N, R_1]$	$\omega_{N,p}\ \omega_{2,p}$ $\sigma[R_N, R_2]$	$\omega_{N,p}\ \omega_{j,p}$ $\sigma[R_N, R_j]$	$\omega_{N,p}\ \omega_{N-1,p}$ $\sigma[R_N, R_{N-1}]$	$\omega_{N,p}\ \omega_{N,p}$ $\sigma[R_N, R_N]$

a. Notice that there are N^2 terms.
b. The diagonal elements are the variance terms since $\sigma^2[R_i(t)] = \sigma[R_i(t), R_i(t)]$; so there are N variance terms and (N-1)N covariance terms.

c. Notice that this formula specializes to the formula used above for the two asset case:

$$\sigma^2[R_p(t)] = \omega_{1,p}^2 \, \sigma[R_1(t)]^2 + \omega_{2,p}^2 \, \sigma[R_2(t)]^2 + 2 \, \omega_{1,p} \, \omega_{2,p} \, \sigma[R_1(t), \, R_2(t)]$$

	1	2
1	$\omega_{1,p} \ \omega_{1,p}$ $\sigma[R_1, R_1]$	$\omega_{1,p} \ \omega_{2,p}$ $\sigma[R_1, R_2]$
2	$\omega_{2,p} \ \omega_{1,p}$ $\sigma[R_2, R_1]$	$\omega_{2,p} \ \omega_{2,p}$ $\sigma[R_2, R_2]$

III. Effect of Diversification.

A. Consider an equal weighted portfolio (So $\omega_{i,p} = 1/N$ for all i.). For example, when N=2, an equal weighted portfolio has 50% in each asset.

B. Suppose all assets have the same $E[R] = \overline{R}$ and $\sigma[R] = \sigma$ and have returns which are uncorrelated. Then

1. N=2:

$E[R_p(t)] = ½\ E[R_1(t)] + ½\ E[R_2(t)] = \overline{R}.$

$\sigma[R_p(t)]^2 = (½)^2\ \sigma[R_1(t)]^2 + (½)^2\ \sigma[R_2(t)]^2 = ½\ \sigma^2.$

2. N=3:

$E[R_p(t)] = a\ E[R_1(t)] + a\ E[R_2(t)] + a\ E[R_3(t)] = \overline{R}.$

$\sigma[R_p(t)]^2 = (a)^2\ \sigma[R_1(t)]^2 + (a)^2\ \sigma[R_2(t)]^2 + (a)^2\ \sigma[R_3(t)]^2 = a\ \sigma^2.$

3. Arbitrary N:

$E[R_p(t)] = \overline{R}.$

$\sigma[R_p(t)]^2 = \sigma^2/N.$

4. As N increases:
   a. the variance of the portfolio declines to zero.
   b. the portfolio's expected return is unaffected.
5. This is known as the effect of diversification.

C. Suppose assets have non-zero covariances and differing expected returns and standard deviations.

1. Formulas for expected portfolio return and standard deviation can be written:

$E[R_p(t)]$ = *average expected return*

$\sigma^2[R_p(t)] = \frac{1}{N}$ *average variance* + $(1 - \frac{1}{N})$ *average covariance*

where

$$\text{average expected return} = \frac{1}{N}\sum_{i=1}^{N} E[R_i(t)]$$

$$\text{average variance} = \frac{1}{N}\sum_{i=1}^{N} \sigma[R_i(t)]^2$$

$$\text{average covariance} = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} \sigma[R_i(t),\ R_j(t)].$$

2. As N increases:
   a. Expected portfolio return is unaffected.
   b. Variance of portfolio return:
      (1) First term (the unique/ firm specific/ diversifiable/ unsystematic risk) goes to zero.
      (2) Second term (the market/ systematic/ undiversifiable risk) remains.
         (a) When the assets are uncorrelated (the case above in III.B), this second term is zero.

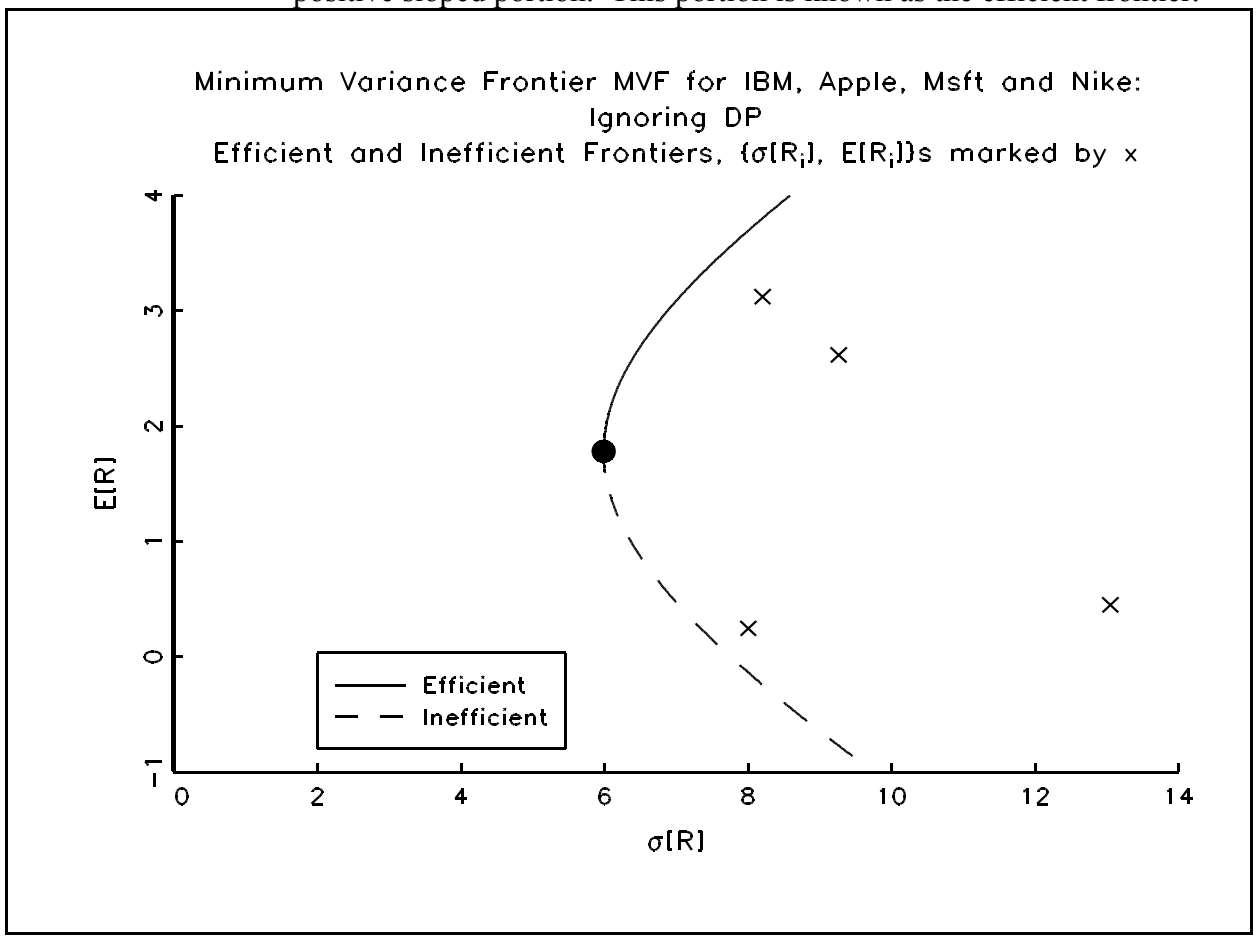
IV. Opportunity Set - N Risky Assets.

A. Set of Possible Portfolios.

1. No longer a curve as in the two asset case.
2. Instead, a set of curves.

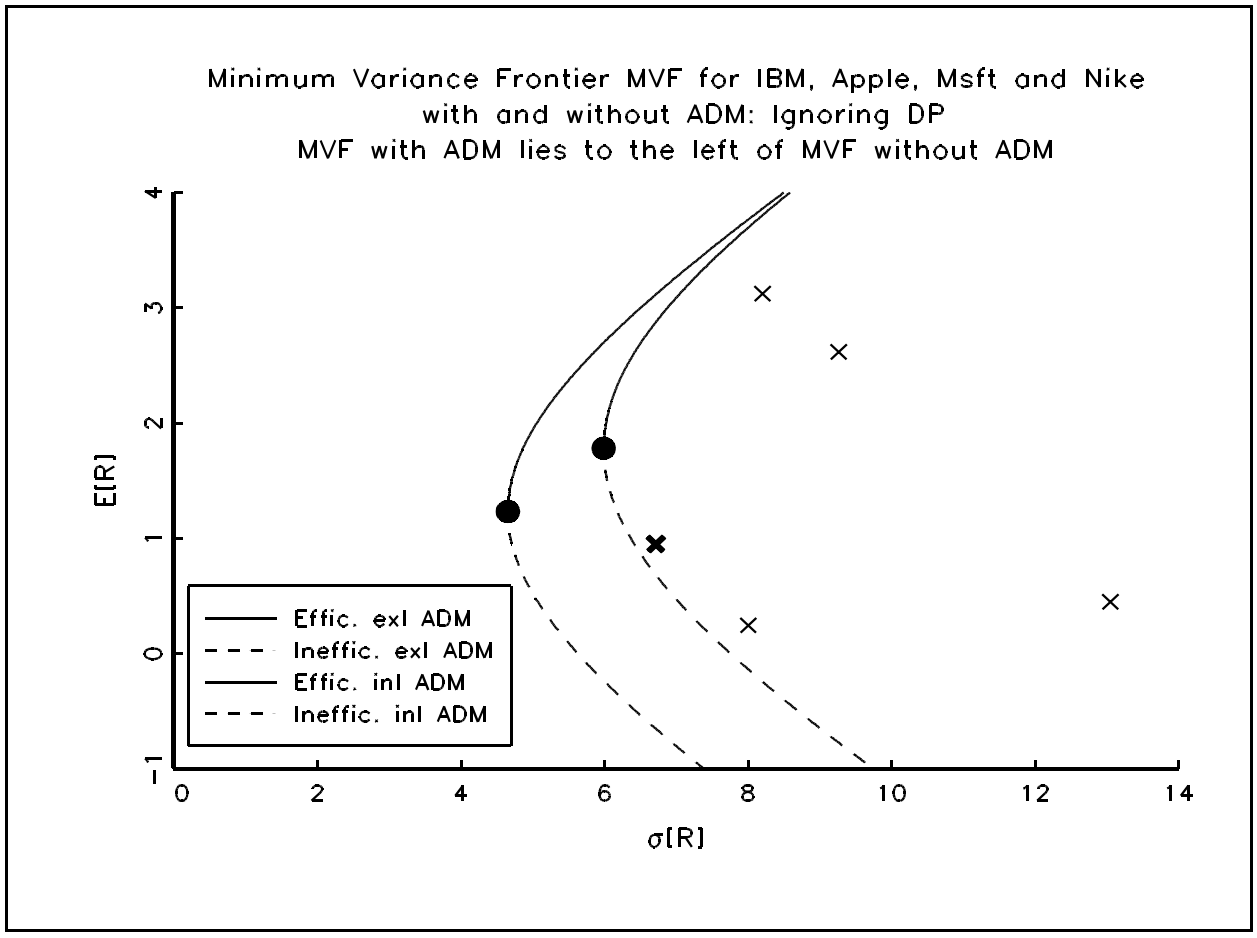
B. Minimum Variance Frontier.

1. Since individuals are risk averse, can restrict attention to the set of portfolios with the lowest variance for a given expected return.
2. This curve is known as the minimum variance frontier (MVF) for the risky assets.
3. Every other possible portfolio is dominated by a portfolio on the MVF (lower variance of return for the same expected return).
4. Example 2 (cont): Ignoring DP. The basic shape of the MVF is the same as the MVF for 4 individual stocks in this example (IBM, Apple, Microsoft and Nike) which is graphed below.
5. Further, risk averse individuals would never hold a portfolio on the negative sloped portion of the MVF; so can restrict attention to the positive sloped portion. This portion is known as the efficient frontier.

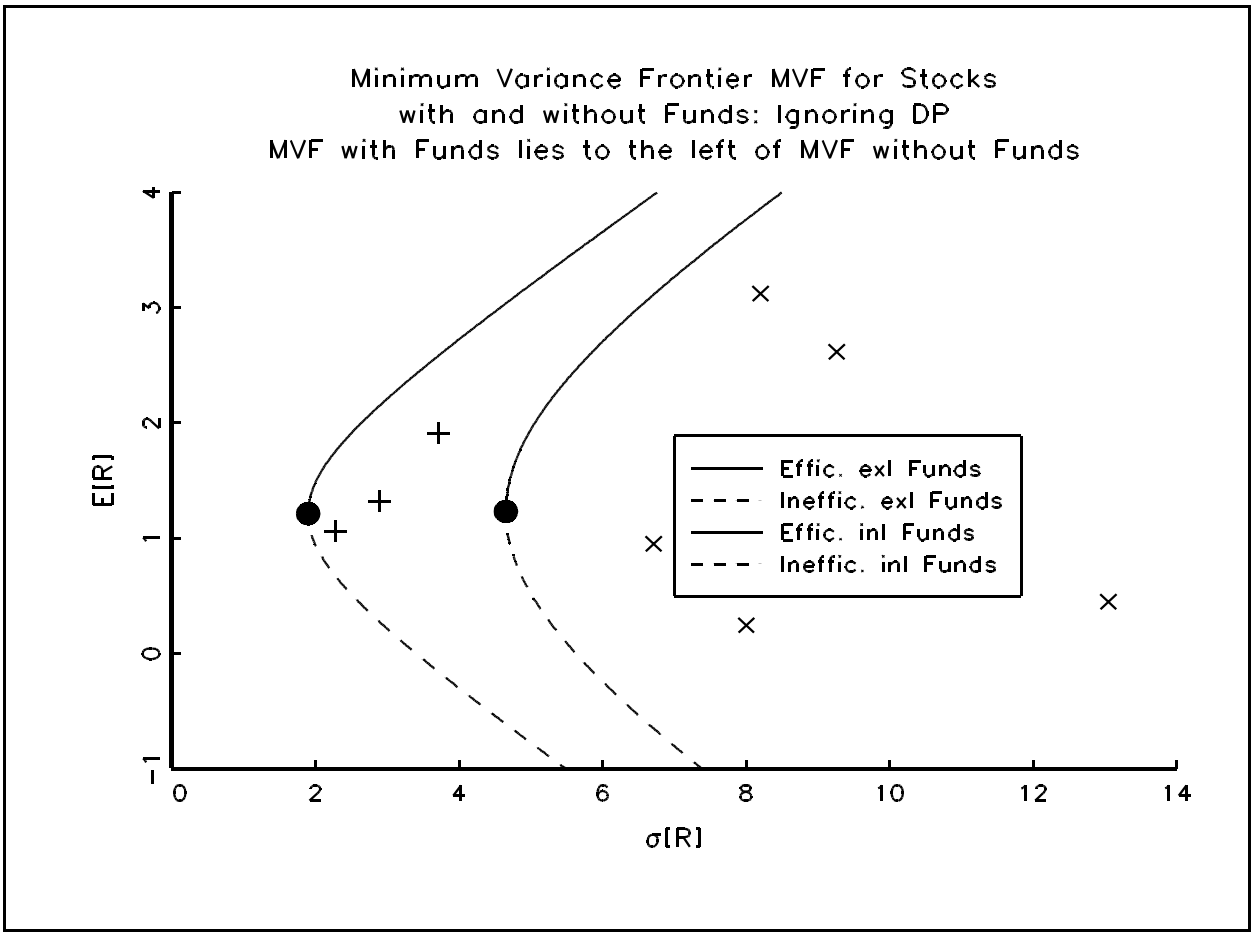


C. Adding risky assets.

1. Adding risky assets to the opportunity set always causes the minimum variance frontier to shift to the left in {σ[R],E[R]} space. Why?
   a. For any given E[R], the portfolio on the MVF for the subset of risky assets is still feasible using the larger set of risky assets.
   b. Further, there may be another portfolio which can be formed from the larger set and which has the same E[R] but an even lower σ[R].
2. Example 2 (cont): Ignoring DP. MVF for IBM, Apple, Microsoft, Nike and ADM is to the left of the MVF for IBM, Apple, Microsoft and Nike excluding ADM. This happens even though ADM has an {σ[R],E[R]) denoted by ✖ which lies to the right of the MVF for the 4 stocks excluding ADM.



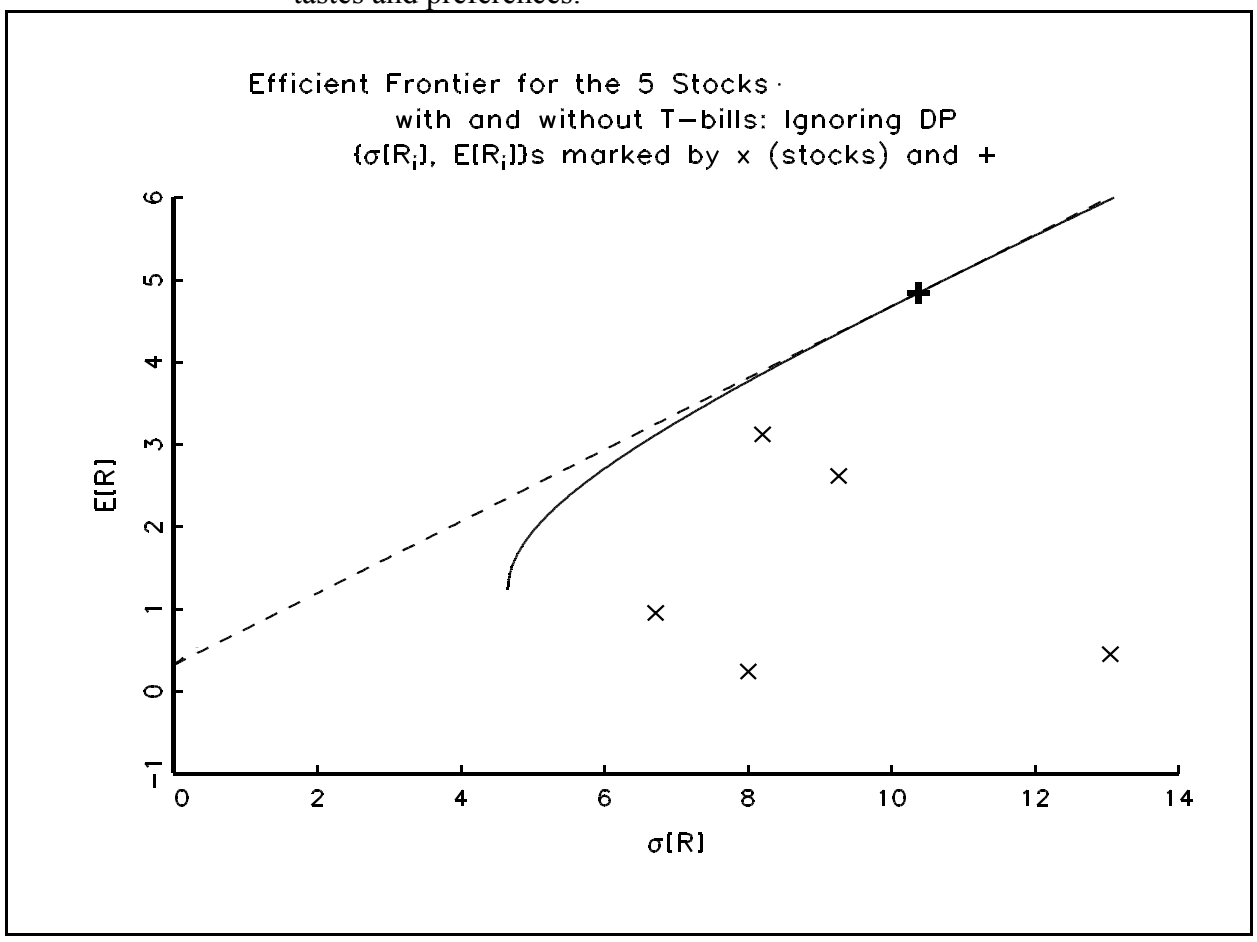
3. Example 2 (cont): Ignoring DP. MVF for all 8 assets (including the 3 funds or portfolios: small firm fund, S&P 500 fund and long-term government bonds fund) is to the left of the MVF for the 5 individual stocks (IBM, Apple, Microsoft, Nike and ADM).



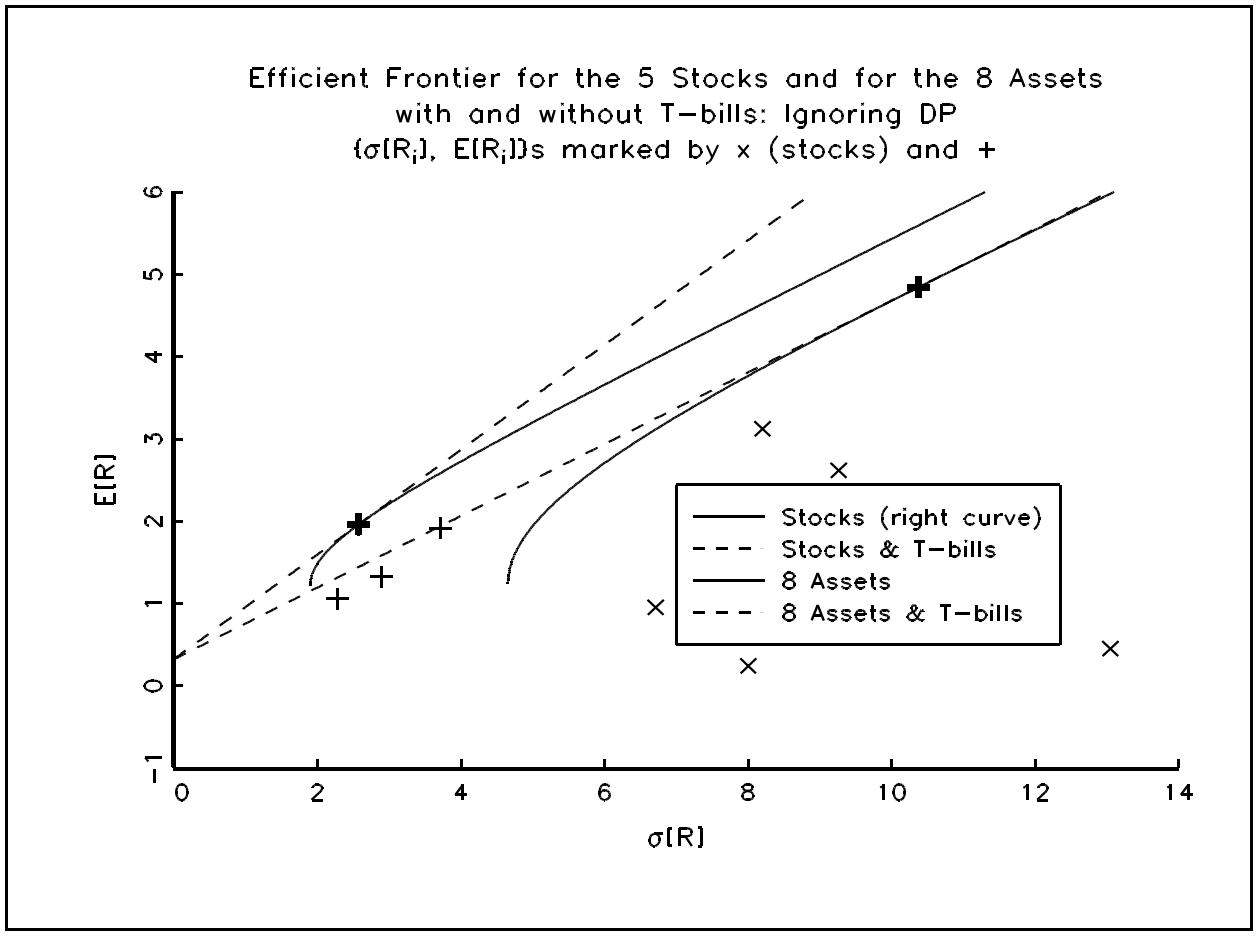
V. Portfolio Choice- N Risky Assets and a Riskless Asset

A. The analysis for the two risky asset and a riskless asset case applies here.

1. Any risk averse individual combines the riskless asset with the risky portfolio whose Capital Accumulation Line has the highest slope.
2. That risky portfolio is on the efficient frontier for the N risky assets and is known as the tangency portfolio (✚): calculating the weights of assets in the tangency portfolio can be performed via computer.
3. All risk averse individuals want to hold this tangency portfolio in combination with the riskless asset. The associated Capital Accumulation Line is the efficient frontier for the N risky assets and the riskless asset.
4. Only the weights of the tangency portfolio and the riskless asset in an individual's portfolio depend on the individual's tastes and preferences.
5. Example 2 (cont): Ignoring DP. If individuals can form a risky portfolio from the 5 individual assets and combine that risky portfolio with T-bills, then all individuals will hold ✚ as their risky portfolio. The weights of ✚ and T-bills in an individual's portfolio will depend on that individual's tastes and preferences.



B. Adding risky assets to the set of available risky assets:
   1. shifts the MVF for the risky assets to the left.
   2. allows investors access to a CAL with a higher slope.
   3. increases the utility of any individuals (in the absence of transaction costs).
   4. Example 2 (cont): Ignoring DP. The slope of the CAL available using all 8 assets is higher than that for the CAL available using only the 5 individual stocks.



C. Transaction Costs.

1. When the transaction costs associated with forming portfolios increase with the number of assets in the portfolio, there may be some optimal number of assets to have in the portfolio.
2. In this case, assets are added to the portfolio until the benefits from adding one more asset are offset by the associated increase in transactions costs.
3. Example 2 (cont): Ignoring DP. If an investor has used the 3 funds and T-bills to form a portfolio, the benefit from adding the 5 individual stocks appears small (see the graph below). If the investor faces significant fixed costs to start trading individual stocks (find a broker, open a brokerage acount, ...) then the individual may prefer not to trade individual stocks.

