IMPERIAL COLLEGE LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2010

This paper is also taken for the relevant examination for the Associateship.

M4A42

Applied Stochastic Processes

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let $W(t)$ be a standard one dimensional Brownian motion.

   (i) Let $c > 0$ arbitrary and define the stochastic processes

   $$X(t) = \frac{1}{\sqrt{c}} W(ct),$$

   and

   $$Y(t) = W(c+t) - W(c).$$

   Show that $\{X(t),\, t \geq 0\} = \{W(t),\, t \geq 0\}$ and $\{Y(t),\, t \geq 0\} = \{W(t),\, t \geq 0\}$ in law.

   (ii) Calculate $\mathbb{E}e^{iW(t)}$ and $\mathbb{E}e^{i(W(t)+W(s))}$, $t, s, \in (0, +\infty)$.

   (iii) Let $B(t) = (1-t)W\left(\frac{t}{1-t}\right)$ for $t \in (0,1)$. Calculate the probability distribution function of $B(t)$.

2. (i) Let $Y_0, Y_1, \ldots$ be a sequence of independent, identically distributed random variables and consider the stochastic process $X_n = Y_n$. Let f be such that $\mathbb{E}(f(Y_0))^2 < +\infty$. Show that

   $$\lim_{N \to +\infty} \mathbb{E} \left| \frac{1}{N} \sum_{j=0}^{N-1} f(X_j) - \mathbb{E} f(Y_0) \right|^2 = 0.$$

   (ii) Let $Y(t)$ be a mean zero stationary Gaussian process with autocorrelation function

   $$\mathbb{E}(Y(t)Y(s)) = e^{-|t-s|}.$$

   Define

   $$X(t) = \int_0^t Y(s)\, ds. \tag{1}$$

   Show that $X(t)$ is a mean zero Gaussian process and calculate its mean and covariance $\mathbb{E}(X(t)X(s))$.

3. Let X_t be a diffusion process in $\mathbb{R}$ with drift and diffusion coefficients $a(x) = a$ and $b(x) = \sigma^2$. Assume that the process starts at x_0, $X(0) = x_0$.

   (i) Write down the generator of the process $X(t)$, the backward and forward (Fokker-Planck) Kolmogorov equations and the corresponding stochastic differential equation.

   (iii) Solve the stochastic differential equation and calculate the mean and variance of X_t.

   (ii) Solve the initial value problem for the forward Kolmogorov (Fokker-Planck) equation to calculate the transition probability density $p(x, t|x_0, 0)$.

   (iv) Consider the same diffusion process in the interval $[0, 1]$ with periodic boundary conditions. Show that this process is an ergodic Markov process and calculate the invariant distribution.

4. Let X_t be a diffusion process in $[0,1]$ with drift and diffusion coefficients $a(x) = 0$ and $b(x) = 1$ and with reflecting boundary conditions. Assume that the process starts at x_0, $X(0) = x_0$.

   (i) Write down the generator of the process X_t and the backward and forward (Fokker-Planck) Kolmogorov equations.

   (ii) Solve the initial-boundary value problem for the forward Kolmogorov (Fokker-Planck) equation to calculate the transition probability density $p(x, t|x_0, 0)$.

   (iii) Show that the process is ergodic and find the invariant distribution $p_s(x)$.

   (iv) Calculate the stationary autocorrelation function

$$\mathbb{E}(W(t)W(0)) = \int_0^1 \int_0^1 x x_0 p(x, t|x_0, 0) p_s(x_0)\, dx dx_0.$$