King Fahd University of Petroleum and Minerals
Department of Mathematics

Math 550 Linear Algebra (Term 261)

Exam 1 (Duration = **2 hours 30 minutes**)

Problem 1. Let V be the vector space over $\mathbb{C}$ of all functions from $\mathbb{R}$ into $\mathbb{C}$. Let $f_1 = 1$, $f_2 = e^{ix}$, $f_3 = e^{-ix}$, $g_1 = 1$, $g_2 = \cos(x)$, and $g_3 = \sin(x)$. Set $B := \{f_1, f_2, f_3\}$ and $G := \{g_1, g_2, g_3\}$.

(1) Prove that B is linearly independent.
(2) Find an invertible matrix $P = (p_{ij})$ such that $g_j = \sum_{i=1}^{3} p_{ij} f_i$, for $j = 1, 2, 3$.
(3) Find P^{-1}.
(4) Is B a basis for V? Justify.

Problem 2. Let V be a nonzero finite-dimensional vector space over a field F and let T be a linear operator on V. Let f and p denote the characteristic polynomial and the minimal polynomial of T, respectively.

(1) Let $\lambda \in F$. Prove that $p(\lambda) = 0$ if and only if $f(\lambda) = 0$.
(2) Assume that T is diagonalizable with distinct characteristic values $\lambda_1, \ldots, \lambda_k$ and let $q := (x - \lambda_1) \cdots (x - \lambda_k)$. Prove that $p = q$.
(3) Prove that p divides f.

Problem 3. Let T be a linear operator and A the matrix associated to T in a standard basis. Let f and p denote the characteristic polynomial and minimal polynomial of T, respectively. Find f and p for each one of the following cases:

(1) $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ over $\mathbb{R}$ **(2)** $A = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ over $\mathbb{R}$ **(3)** $A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ over $\mathbb{C}$

Problem 4. Let V be an F-vector space of dimension n and W a subspace of dimension $k \leq n - 1$.

(1) Prove that W is equal to a finite intersection of hyperspaces in V.
(2) Prove that every linear functional $f : W \to F$ extends to V $\left(\text{i.e., } \exists\, \overline{f} \in V^\star \text{ such that } \overline{f}\big|_W = f\right)$.

Problem 5. Prove the following:

(1) Let V be a two-dimensional vector space over a field F and let T be a linear operator on V such that $T^2 = 0$. Prove that either $T = 0$ or there exists an ordered basis B for V such that $[T]_B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
(2) Every nonzero 2×2 complex matrix is similar to either $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $\begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$, for some $a, b \in \mathbb{C}$.