642:621, Mathematical Finance: Lecture 2 Notes

I. Pricing in the multi-period, binomial model.

A. The multi-period, binomial model; reviewed.

The multi-period, binomial model for one risky asset and a money market account was defined in the Lecture 1 Notes. What follows is a summary of the definition. Let n denote the number of periods and let h denote the length of each period. The trading times of the model are then $t_0 = 0$, $t_1 = h$, $t_2 = 2h, \ldots, t_n = nh = T$. The set Ω, whose elements describe the possible future market histories (also called market paths), is the set $\{-1, 1\}^n$ of all sequences $\omega = (\omega_1, \ldots, \omega_n)$, where each ω_i is either -1 or 1 and indicates which one of two possible movements the market makes; if $S(t_j)(\omega)$ denotes the price of the risky asset, $\omega_i = -1$ implies $S(t_i)(\omega) = dS(t_{i-1})(\omega)$ and $\omega_i = 1$ implies $S(t_i)(\omega) = uS(t_{i-1})(\omega)$. Here $0 < d < u$ are fixed parameters, the same for all time periods. The risk-free money market account is specified by $B(t_i) = (1 + rh)^i$; $B(t_i)$ denotes the value at t_i of an investment of one dollar at $t_0 = 0$ at the risk-free rate r, compounded period by period.

In equation (2) of the Notes to Lecture 1, it was shown that

$$S(t_k)(\omega) = (ud)^{i/2} \left(\frac{u}{d}\right)^{(1/2)\sum_1^k \omega_i} S(0). \quad \text{for any } k \geq 0 \text{ and any } \omega.$$

A simple calculation shows this may be written

$$S(t_k)(\omega) = (u)^{(1/2)\sum_1^k (\omega_i + 1)} (d)^{(1/2)\sum_1^k (1-\omega_i)} S(0). \tag{1}$$

It is seen that $S(t_k)(\omega)$ depends only on $(\omega_1, \ldots, \omega_k)$, the history of market movements up to period k.

It shall be assumed henceforth that $d < 1 + rh < u$. As was argued in the Notes to Lecture 1, this implies the model is arbitrage-free.

B. Contingent claims and the question of pricing.

A contingent claim with payoff at time T is specified by a function $H(\omega)$ on the set Ω: for each $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$, $H(\omega)$ is what the contingent claim pays at T if the market has followed the path ω.

It will be assumed that a contingent claim can be traded at all times. The price of the contingent claim at time t_i as a function of ω is denoted $V(t_i)(\omega)$. Of course, since the payoff is $H(\omega)$

$$V(T)(\omega) = H(\omega) \quad \text{for all } \omega. \tag{2}$$

The price $V(t_i)(\omega)$ can only depend on the market history $(\omega_1, \ldots, \omega_i)$ up to time t_i, so really we can and will write $V(t_i)(\omega) = V(t_i)(\omega_1, \ldots, \omega_i)$.

The question we want to answer is: what should the price process $V(t_i)(\omega)$, $0 \leq i \leq n$, $\omega \in \Omega$, be in order that the market remains arbitrage-free? It turns out that the no-arbitrage condition implies a unique price for all t_i and ω. It will be shown how to calculate this price.

C. Recursion for $V(t)(\omega)$.

Fix a k and suppose, hypothetically, that the prices $V(t_{k+1})(\omega_1, \ldots, \omega_k, \omega_{k+1})$, for all $(\omega_1, \ldots, \omega_k, \omega_{k+1})$, are known for the contingent claim at the end of the next period. Now suppose that we have arrived at time t_k and have observed the market history $(\omega_1, \ldots, \omega_k)$ up to that point. There are then two possibilities for the market history up to t_{k+1}; either

$$(\omega_1, \ldots, \omega_k, -1) \quad \text{or} \quad (\omega_1, \ldots, \omega_k, 1).$$

Owning the contingent claim is equivalent to owning a claim that pays $V(t_{k+1})(\omega_1, \ldots, \omega_k, -1)$ at time t_{k+1} in the first case, or $V(t_{k+1})(\omega_1, \ldots, \omega_k, 1)$ in the second. We know from the analysis of the one-period problem, that one can achieve an arbitrage over the $(k+1)^{\text{st}}$ unless the price of the claim at t_k and $(\omega_1, \ldots, \omega_k)$ is

$$V(t_k)(\omega_1, \ldots, \omega_k) = \frac{1}{1+rh}\Big[\tilde{q}V(t_{k+1})(\omega_1, \ldots, \omega_k, -1) + \tilde{p}V(t_{k+1})(\omega_1, \ldots, \omega_k, 1)\Big] \quad (3)$$

where

$$\tilde{q} = \frac{u - (1+rh)}{u-d}, \quad \tilde{p} = \frac{1+rh-d}{u-d}.$$

Equation (3) is a backward-in-time recursive formula, called a *dynamic programming equation*, that the price of the contingent claim must satisfy at all times t_k, $0 \leq k \leq n-1$ in order that there be no arbitrage.

D. Solving for the contingent claim price.

Equations (2) and (3) uniquely determine the price, at all times and all ω, of a contingent claim paying $H(\omega)$ at time $T = nh$. By applying (3) for $k = n-1$,

$$\begin{aligned} V(t_{n-1})(\omega_1, \ldots, \omega_{n-1}) &= \frac{1}{1+rh}\Big[\tilde{q}V(T)(\omega_1, \ldots, \omega_{n-1}, -1) + \tilde{p}V(T)(\omega_1, \ldots, \omega_{n-1}, 1)\Big] \\ &= \frac{1}{1+rh}\Big[\tilde{q}H(\omega_1, \ldots, \omega_{n-1}, -1) + \tilde{p}H(\omega_1, \ldots, \omega_{n-1}, 1)\Big]. \end{aligned}$$

This gives the price at time t_{n-1} for all states of the market. Now the price at time t_{n-2} can be found from $V(t_{n-1})$ by using (3) with $k = n-2$. Working backwards this way step by step, one finally is able to compute the price of the contingent claim for all times and all states.

E. Special case for a claim written on the asset price.

For a general contingent claim, computing $V(t_i)(\omega)$ by iterating (3) is computationally heavy for even moderate sized n, because the number of possible market histories $(\omega_1, \ldots, \omega_n)$ is 2^n, which grows rapidly with n. However if the payoff H is a function of the price of the underlying, in other words, if

$$V(T)(\omega) = H(\omega) = C(T, S(T)(\omega)),$$

for some function $C(T, s)$, the price of the contingent claim will be a function of the price of the underlying at all times, and the recursion becomes simpler. The following statement makes this claim precise.

Theorem 1 *Let $T = nh$. If $V(T)(\omega) = C(T, S(T)(\omega))$, then there is a function $C(t_k, s)$, $1 \le k < n$, such that*

$$V(t_k)(\omega) = C(t_k, S(t_k)(\omega)).$$

The function $C(t_k, s)$ is defined for each vertex (t_k, s) on the recombining tree representing the market. (Recall that at the end of period k there are $k+1$ possible prices given by $\{u^\ell d^{k-\ell} S(0);\ 0 \le \ell \le k\}$—these represent the vertices (t_k, s).) $C(t_k, s)$ satisfies the backward-in-time recursion: for $0 \le k < n$,

$$C(t_k, s) = \frac{1}{1+rh}\Big[\tilde{q}C(t_{k+1}, ds) + \tilde{p}C(t_{k+1}, us)\Big]. \tag{4}$$

The proof is by backward induction. Show using (3) that if we know $V(t_{k+1})(\omega) = C(t_{k+1}, S(t_{k+1})(\omega))$ is true then $V(t_k, \omega) = C(t_k, S(t_k, \omega))$, where $C(t_k, s)$ is found from $C(t_{k+1}, s)$ by (4). In this way, working backward from $k+1 = n$, one proves the validity of the theorem for all k. A detailed proof is left as an exercise.

Calculation is considerably simpler in the case described by Theorem 1. The price of the claim only has to be computed for each price in the recombining tree, which has on the order of n^2 vertices for an n-period model, and one can carry out the backward recursion directly on the tree itself. For example, consider the recombining tree in the following figure:

Suppose we want to compute the price of a European call on this asset, when the strike is K. We know the values of the call on the right-most vertices corresponding to $T = t_3$—they are just the payoffs $(S(T) - K)^+$. Now step back one period to t_2. For example, consider the middle vertex, corresponding to time t_2. The price at this vertex is $udS(0)$. Using the recursion (3), the price of the call at time t_2 for this value of the price is

$$C(t_2, udS(0)) = \frac{1}{1+rh}\Big[\tilde{q}(ud^2S(0) - K)^+ + \tilde{p}(u^2dS(0) - K)^+\Big].$$

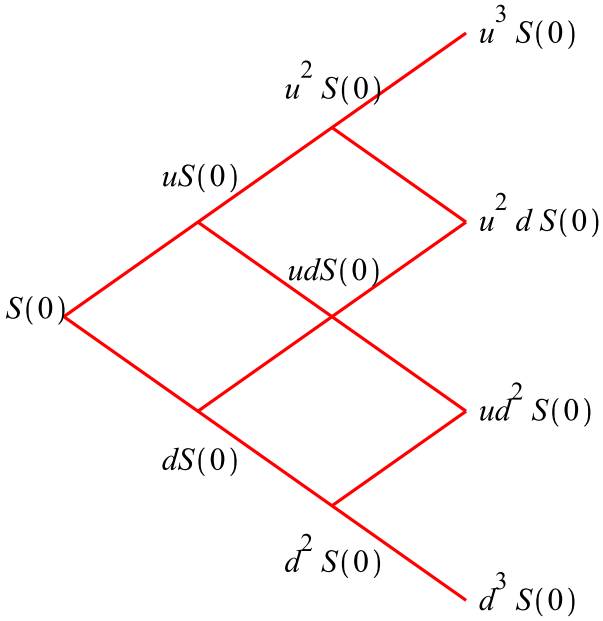


Figure 1: Binomial tree, three periods.

The values appearing on the right, are just the option prices at the two vertices that can be reached from the vertex corresponding to $udS(0)$.

In a similar way, one computes the prices at all the vertices corresponding to t_2. Once these are known, one can step back to time t_1 and finally to time $t_0 = 0$.

The theory discussed here is all explained in volume I, Chapter 1 of Shreve.

F. An explicit formula for $V(t_k)$

It is possible to find an explicit formula for the solution to (2) and (3) in the general case. To state this, it will be convenient to use the notation $\{-1,1\}^k$ for the set of all sequences $\eta = (\eta_1, \ldots, \eta_k)$ of length k containing only -1's and 1's. For any integer j and sequence $\eta = (\eta_1, \ldots, \eta_j) \in \{-1,1\}^j$, define

$$N_j(\eta) := \text{number of 1's in } \eta = (\eta_1, \ldots, \eta_j) \in \{-1,1\}^j.$$

For example $N_4(1,1,-1,1) = 3$, $N_6(1,-1,-1,-1,1,-1) = 2$, etc. If we think of η as a market history of length j, $N_j(\eta)$ counts the number of times the market moves up. Clearly $j - N_j(\eta)$ is then the number of -1's in $\eta = (\eta_1, \ldots, \eta_j) \in \{-1,1\}^j$, or the number of times the market moves down.

One can check that

$$N_j(\eta) = \frac{1}{2}\sum_1^j (1+\eta_i),$$

because, for each i, $\frac{1+\eta_i}{2} = 1$ if $\eta_i = 1$ and equals 0 if $\eta_i = -1$.

Finally, one more notational convention is useful. Suppose $0 \leq k < n$ and $\eta = (\eta_1, \ldots, \eta_{n-k}) \in \{-1, 1\}^{n-k}$. Then we let

$$H(\omega_1, \ldots, \omega_k, \eta) \text{ denote } H(\omega_1, \ldots, \omega_k, \eta_1, \ldots, \eta_{n-k}).$$

Theorem 2 *The solution to (2) and (3) is,*

$$V(t_k)(\omega_1, \ldots, \omega_k) = \frac{1}{(1+rh)^{n-k}} \sum_{\eta \in \{-1,1\}^{n-k}} \tilde{q}^{n-k-N_{n-k}(\eta)} \tilde{p}^{N_{n-k}(\eta)} H(\omega_1, \ldots, \omega_k, \eta) \quad (5)$$

In particular,

$$V(0) = \frac{1}{(1+rh)^n} \sum_{\eta \in \{-1,1\}^n} \tilde{q}^{n-N_n(\eta)} \tilde{p}^{N_n(\eta)} H(\eta) \quad (6)$$

To understand this formula it helps to write out a case. For example,

$$\begin{aligned} V(t_{n-2})(\omega_1, \ldots, \omega_{t-2}) &= \frac{1}{(1+rh)^2} \big[\tilde{q}^2 H(\omega_1, \ldots, \omega_{t-2}, -1, -1) + \tilde{q}\tilde{p} H(\omega_1, \ldots, \omega_{t-2}, -1, 1) \\ &\qquad \tilde{q}\tilde{p} H(\omega_1, \ldots, \omega_{t-2}, 1, -1) + \tilde{p}^2 H(\omega_1, \ldots, \omega_{t-2}, 1, 1) \big] \end{aligned}$$

Formula (5) can be proved by backward induction. You should understand this idea and follow the two steps below. First, its validity for $k = n-1$ can be checked directly. Note that $\{-1, 1\}^1 = \{-1, 1\}$, so $\eta \in \{-1, 1\}$ is just a number in $\{-1, 1\}$. Also, note that if $\eta \in \{-1, 1\}$, $N_1(\eta) = 1$ if $\eta = 1$ and $N_1(\eta) = 0$ if $\eta = -1$. Thus, when $k = n-1$ equation (5) is just

$$\begin{aligned} V(t_{n-1})(\omega_1, \ldots, \omega_{t-1}) &= \frac{1}{1+rh} \sum_{\eta \in \{-1,1\}} \tilde{q}^{1-N_1(\eta)} \tilde{p}^{N_1}(\eta) H(\omega_1, \ldots, \omega_{t-1}, \eta) \\ &= \frac{1}{(1+rh)} \big[\tilde{q} H(\omega_1, \ldots, \omega_{t-1}, -1) + \tilde{p} H(\omega_1, \ldots, \omega_{t-2}, 1) \big] \end{aligned}$$

But the right-hand side is exactly the formula for $V(t_{n-1})$ given by the recursion (3)—see paragraph C.

Now let us go to $k = n-2$. From first using (3) and then the expression for $V(t_{n-1})$ we just derived

$$\begin{aligned} V(t_{n-2})(\omega_1, \ldots, \omega_{n-2}) &= \frac{1}{1+rh} \big[\tilde{q} V(t_{n-1})(\omega_1, \ldots, \omega_{n-2}, -1) + \tilde{p} V(t_{n-1})(\omega_1, \ldots, \omega_{n-2}, 1) \big] \\ &= \frac{1}{1+rh} \big[\tilde{q} \frac{1}{1+rh} [\tilde{q} H(\omega_1, \ldots, -1, -1) + \tilde{p} H(\omega_1, \ldots, -1, 1)] \end{aligned}$$

$$
\begin{aligned}
& + \tilde{p}\frac{1}{1+rh}[\tilde{q}H(\omega_1,\ldots,1,-1) + \tilde{p}H(\omega_1,\ldots,1,1)]\Big] \\
= \quad & \frac{1}{(1+rh)^2}\Big[\tilde{q}^2 H(\omega_1,\ldots,\omega_{t-2},-1,-1) + \tilde{q}\tilde{p}H(\omega_1,\ldots,\omega_{t-2},-1,1) \\
& \tilde{q}\tilde{p}H(\omega_1,\ldots,\omega_{t-2},1,-1) + \tilde{p}^2 H(\omega_1,\ldots,\omega_{t-2},1,1)\Big]
\end{aligned}
$$

This last expression is exactly what (5) gives for t_{n-2}, as we calculated in the example above. So this verifies the case $k = n - 2$.

The general induction proof requires showing that if the formula (5) is true for $k + 1$, it is true for k. This proof is optional reading. A detailed treatment may be found in Shreve, Volume I, Chapter 1.